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ELLIPTIC CYLINDER AND SPHEROIDAL WAVE FUNCTIONS

STRATTON – MORSE – CHU – HUTNER

ELLIPTIC CYLINDER AND SPHEROIDAL WAVE FUNCTIONS

INCLUDING TABLES OF SEPARATION CONSTANTS AND COEFFICIENTS

Mat Tabellen

BY

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FOREWORD

In the past fifteen years the wave equation has come to have a steadily increasing importance in theoretical physics. In addition to the increase in practical importance of the classical fields of acoustics and electromagnetic theory, the new and fundamental field of wave mechanics has come into being. The developments in this field have shown that a basic understanding of the structure of matter and of particle dynamics can only be obtained through a study of a wave equation. Thus, both old fields of application of wave theory and basically important new fields have indicated an expanding need for an improvement in the techniques for the calculation of solutions of the wave equation.

For the solution of many general problems involving wave motion, general solutions of the wave equation can be used. For instance, in many cases a sum, or integral, of plane waves may be used to compute general behavior; although in many such computations serious questions of convergence of the series may arise. Recent calculations in quantum electrodynamics are typical examples of the plane wave summation technique, and also of the difficulties this technique encounters. There is increasing need, however, for the solution of specific and detailed problems rather than general cases; and for this work representations of the wave function must be found which are better and more rapidly convergent than the plane wave aggregates.

This need for better representations is particularly apparent with wave problems involving boundary conditions on finite boundaries. In such cases, usually the only practicable method of numerical solution involves

the separation of the wave equation in the particular co-ordinate system which corresponds to the boundary. Thus the partial differential equation is reduced to a sequence of ordinary differential equations, one for each co-ordinate. This method is exceedingly powerful in allowing specific boundary value problems to be solved in detail; but it is severely limited in its scope, for the wave equation is separable in only a few Euclidian co-ordinate systems. The work of Robertson[1] and of Eisenhart[2] has shown that only eleven different co-ordinate systems allow separation of the scalar wave equation in three dimensions. Some further work has indicated that the limitations for the vector wave equation in this respect are even more binding, for it appears likely that only five different tri-dimensional co-ordinate systems allow vector solutions of the wave equation which can be separated in such a manner as to satisfy boundary conditions on the co-ordinate surfaces.

This limitation of the applicability of the separation technique is a serious one and it appears urgent that further fundamental research be done in the development of other methods of solution. Nevertheless, from the point of view of the application of wave theory to physics and to engineering, the possible usefulness of the separation technique is far from having been exhausted. In order to apply this technique to a particular problem the solutions of the ordinary differential equations arising from the separation must be obtained, their mathematical properties must be thoroughly explored, and tables of values of the solutions must be prepared. Only after these three steps have been taken can numerical solutions to specific problems be computed without great difficulty. This state of affairs has been reached at present for only three of the eleven co-ordinate systems: rectangular, circular cylinder, and

spherical co-ordinates.

The ordinary differential equations arising from the separation of the wave equation, in the eleven cases mentioned, all turn out to be second order and linear, with the singular points of the equation determined primarily by the geometrical properties of the particular co-ordinate system. For instance, the equations arising from the separation into rectangular co-ordinates have a single irregular singular point at infinity; whereas some of the equations for the circular cylinder and spherical co-ordinates have one regular singular point at zero and one irregular point at infinity, and other equations have three regular singular points. The mathematical properties of the solutions of equations with two and three regular singular points, one irregular, or one regular and one irregular point, have been quite thoroughly studied in the past century. Tables of values of some of the solutions (trigonometric functions, Bessel functions, spherical harmonics, etc.) have been computed and are available in various states of completeness. Thus the practical problem of solutions for plane, cylindrical, or spherical waves has been made as easy as possible for the engineer and applied physicist. The solutions for the other eight separable co-ordinate systems, however, are in no such satisfactory state.

In order of practical importance, the next three co-ordinate systems to be made available for application should be the elliptic cylinder, the prolate spheroidal, and the oblate spheroidal co-ordinates. Solutions of problems involving the radiation and scattering of waves from strips of material, from wires of finite length, and from discs of material, all require the knowledge of the mathematical properties and the numerical values of solutions of the wave

equation for these three co-ordinate systems. The solutions are likewise required for the study of the diffraction of waves through slits and circular openings, the absorption of sound by strips or by circular patches of material, and the behavior of electrons in diatomic molecules, to mention a few cases.

The ordinary differential equations arising from the separation of the wave equation in these three co-ordinate systems have two regular and one irregular singular points. The mathematical properties of the solutions of equations of this sort have been studied for some time. Mathieu[3] began the study of their application, with his classical study of the vibrations of an elliptical membrane; since that time Ince[4], Humbert[5], Strutt[6] and many others have made contributions. A consolidation of the knowledge of these solutions has not been completed as yet, however. In particular, a standard form of solution and a standard notation for the solutions has not yet been agreed upon. Since such agreement in other cases has usually accompanied or followed the publication of reasonably complete tables of solutions, it is hoped that the present volume will help to make agreement possible for elliptic cylinder and spheroidal functions.[7]

The present volume defines certain standard forms of the solution which turn out to be of use in practical problems, and it displays a collection of formulas giving the important mathematical properties of these functions. Many of these properties have been discovered and studied earlier. They are given here in terms of the standard form of solution in order to make the outline complete.

In addition to the mathematical properties of the solutions, there is also a set of tables from which values of the solutions can be obtained for the more interesting

ranges of the variables. These tables contain the series coefficients together with some of the separation constants and allied coefficients, and cover the most difficult part of the whole procedure of obtaining numerical solutions. Although it is expected that eventually a larger volume will be published which will give actual numerical values of the solutions themselves, nevertheless, these numerical solutions can be obtained relatively quickly from the present tables of coefficients.

It is to be hoped that the present volume will begin to make it possible for the applied physicist and engineer to handle wave problems in elliptic-cylindrical and spheroidal co-ordinates with approximately the same degree of facility as has been possible previously for rectangular, circular cylindrical, and spherical co-ordinates.

The present tables are the result of a number of years of effort on the part of the authors.[8] Many persons at the Massachusetts Institute of Technology have contributed to the work and the complete list of acknowledgments would be a long one. Miss Pearl Rubenstein helped one of the authors publish a preliminary table of Mathieu functions several years ago. The present tables are to some extent based on these earlier ones. The N. Y. A. has contributed toward the payment of a number of students in order to help in some preliminary calculations. The Department of Physics at the Massachusetts Institute of Technology has contributed computing machines and other equipment, and the Differential Analyzer staff has been of considerable aid in the work. The writers wish to express their appreciation to Professor S. A. Caldwell, in charge of the Massachusetts Institute of Technology Computing Center, for his help in this respect and to Mr. J. R. Killian for his valuable assistance in making possible the final calculations and the publication

of these tables. The authors also wish to express their appreciation of the continuing interest of President K. T. Compton in their lengthy task.

Philip M. Morse

Cambridge, Mass.
October 18, 1941.

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Phil. Mag. (7) 6, 547, 1928. (Contains references to other papers.)

5. Humbert, P.: Fonctions de Lamé et Fonctions de Mathieu. Fasciscule X du Mémorial des sciences mathématiques. Paris: Gauthier-Villars 1926.

6. Strutt, M. J. O.: Lamésche-Mathieusche und Verwandte Funktionen in Physik und Technik. Berlin: Julius Springer 1932. (Contains excellent list of references.)

7. It is desirable to call the reader's attention to the fact that in this volume the notation for the elliptic cylinder functions, and both the notation and the method of normalization for the spheroidal functions are new, and therefore different from those appearing in the previous papers published by any of the authors.

For example, the function $Se^{(1)}_{a,\ell}(c,z)$ in the paper by Stratton, Proc. Nat. Acad. Sci., 21, 51, 1935, is now denoted by $V^{(1)}_{a,\ell}(c,z)$, and the function $Re^{(1)}_{a,\ell}(c,z)$ by $U^{(1)}_{a,\ell}(c,z)$, with similar changes for the solutions of the second kind. In the case of the elliptic cylinder functions, $Se^{(1)}_{-1/2,\ell}(c,z)$ is now represented by $Se^{(1)}_{\ell}(c,z)$, $Re^{(1)}_{-1/2,\ell}(c,z)$ by $Je_{\ell}(c,z)$, and $Re^{(2)}_{-1/2,\ell}(c,z)$ by $Ne_{\ell}(c,z)$, with similar changes for the odd functions.

On the other hand, the formulas in spheroidal coordinates given in the paper by Morse, Proc. Nat. Acad. Sci., 21, 56, 1935, are affected by the change in normalization. Thus equation (14) for the plane wave addition formula becomes:

$$e^{ikX} = 2\sum_{mn} \cos[m(\varphi-\alpha)] \frac{i^{m+n}(2-\delta_{o,m})}{2^m m!\, N_{mn}} \left[\sum_{k=0,1}^{\infty}{}' d^n_k \frac{(k+2m)!}{k!}\right] \times$$

$$S^{(1)}_{mn}(c,\cos\omega)\psi^{(1)}_{mn}(k;\theta,\mu),$$

where $\psi^{(1)}_{mn}(k;\theta,\mu) = S^{(1)}_{mn}(c,\cos\theta)R^{(1)}_{mn}(c,\cosh\mu)$;

the direction of propagation with respect to the z axis is defined by the angles ω and α; and

$$X = \frac{d}{2}\left[\cos\omega\ \cos\theta\ \cosh\mu + \sin\omega\ \sin\theta\ \sinh\mu\ \cos(\varphi-\alpha)\right];$$

$$k = \frac{2\pi}{\lambda}; \quad c = \frac{\pi d}{\lambda}.$$

However, in the elliptic cylinder coordinates, the normalization has remained unchanged. Thus the equations given in Morse's paper are still valid, though there is a slight change in notation. For example,

$$e^{ikY} = \sqrt{8\pi} \sum_n i^n \left[\frac{1}{N_n} Se_n^{(1)}(c,\cos u)\psi_n^{(1)}(k;\varphi,\mu) + \frac{1}{N_n'} So_n^{(1)}(c,\cos u)\chi_n^{(1)}(k;\varphi,\mu) \right],$$

where $\psi_n^{(1)}(k;\varphi,\mu) = Se_n^{(1)}(c,\cos\varphi)Je_n(c,\cosh\mu);$

$\chi_n^{(1)}(k;\varphi,\mu) = So_n^{(1)}(c,\cos\varphi)Jo_n(c,\cosh\mu);$

u is the angle with respect to the x axis of the direction of propagation of the plane wave; and

$$Y = \frac{d}{2}(\cos u \cos\varphi \cosh\mu + \sin u \sin\varphi \sinh\mu).$$

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ERRATA

p. 3 Equation (14): for "$\nabla \times \nabla \times s$" read "$\mu \nabla \times \nabla \times s$."

p. 18 Top of page: for "$T^m_{m+\rho}$" read "$\tau^m_{m+\rho}$."

p. 49 Equation (310): for "$\sqrt{\frac{2}{\pi}}$" read "$\sqrt{\frac{\pi}{2}}$"

p. 54,57 First and second equations below "2. Radial Functions": for "m" in the arguments of the cos and the sin read "ℓ."

p. 56 The equation for λ_ℓ, ℓ = even: for "D^ℓ_1" read "D^ℓ_0."

p. 57 Bottom of page: for "$z = \cos\psi$" read "$z = \cosh\psi$."

p. 58 All E's and L's should have subscripts "ℓ."

p. 62 Third equation from the top: for "$z = \eta = \xi$" read "$z = \eta$ or ξ."

p. 63 Top equation for "dP^m_{m+n}" read "$dP^m_{m\ell}$."

P. 72 Third line: omit the word "negative."
Thirteenth line: for "negative coefficients" read "coefficients with negative subscripts."

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ELLIPTIC AND SPHEROIDAL WAVE FUNCTIONS

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Contents

I. Introduction

1.1 **Scalar Wave Equation.** It is known that the wave equation[1]

[1] The most complete account of earlier researches on the subject matter of this paper is contained in a monograph by M. J. O. Strutt, "Lamésche, Mathieusche, und verwandte Funktionen in Physik und Technik," Springer, 1932. An extensive bibliography is included.

$$\nabla^2 W + k^2 W = 0 \tag{1}$$

may be separated in spheroidal coordinates ξ, η, φ obtained from the cylindrical coordinates r, φ; z through the transformation

$$r = f\sqrt{(\xi^2 \mp 1)(1 - \eta^2)}, \qquad \varphi = \varphi, \qquad z = f\xi\eta, \tag{2}$$

the upper or lower sign being chosen according as the coordinates are of prolate or oblate type. The factors of the resulting Lamé product are then

$$W = (\xi^2 \mp 1)^{\frac{m}{2}} U(\xi)(1 - \eta^2)^{\frac{m}{2}} V(\eta)\Phi(\varphi) \tag{3}$$

and satisfy the ordinary equations

$$\Phi''(\varphi) + m^2\Phi(\varphi) = 0, \tag{4}$$

$$(\xi^2 \mp 1)U'' + 2(m + 1)\xi U' - (b - f^2k^2\xi^2)U = 0, \tag{5}$$

$$(1 - \eta^2)V'' - 2(m + 1)\eta V' + (b \mp f^2k^2\eta^2)V = 0, \tag{6}$$

where b is a separation constant to be determined from conditions of finiteness and periodicity over the surface of a spheroid. Both (5) and (6) can be reduced to the standard form

$$(1 - z^2)U'' - 2(a + 1)zU' + (b - c^2z^2)U = 0. \tag{7}$$

The coordinates of the elliptic cylinder are expressed in terms of rectangular coordinates by the transformation

$$x = f\xi\eta, \qquad y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}, \qquad z = z \tag{8}$$

In this system the scalar wave equation (1) separates into the factors

$$W = U(\xi)V(\eta)Z(z) \tag{9}$$

satisfying the ordinary equations

$$Z'' + \gamma^2 Z = 0, \tag{10}$$

$$(\xi^2 - 1)U'' + \xi U' - [b - f^2(k^2 - \gamma^2)\xi^2]U = 0 \tag{11}$$

$$(1 - \eta^2)V'' - \eta V' + [b - f^2(k^2 - \gamma^2)\eta^2]V = 0. \tag{12}$$

Equations (11) and (12) are obviously special cases of (7) for $a = -\frac{1}{2}$. Their solutions are known generally as Mathieu functions.

To visualize the geometrical significance of certain limiting cases the following will prove helpful. The eccentricity of an ellipse $\xi =$ constant is $e = 1/\xi$. Let l be the length of a semi-major axis. Then the coordinate ξ of an ellipse is

$$\xi = l/f = 1/e, \tag{13}$$

thus appearing to be a pure number, the ratio of semi-major axis to semi-focal distance, or reciprocal of the eccentricity. The limiting case of a circle occurs when $f \to 0$, $\xi \to \infty$ such that the product ξf remains finite and equal to the limiting radius l. Identical relations hold for the spheroids obtained by rotating an ellipse about a principal axis.

1.2 **Vector Wave Equation.** In the theory of elasticity, as well as in electromagnetic theory, one must deal frequently with vector rather than scalar wave functions. Thus the equation governing the propagation of elastic waves in a homogeneous, isotropic medium is

$$\nabla \times \nabla \times \mathbf{s} - (\lambda + 2\mu)\nabla\nabla\cdot\mathbf{s} + \rho\frac{\partial^2 \mathbf{s}}{\partial t^2} = 0, \tag{14}$$

where $\mathbf{s}$ is the relative displacement of a point in the medium, ρ is the density and λ and μ are elastic constants. Likewise an electromagnetic field whose angular frequency is ω can be derived from the Hertz vectors $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ according to the rules[2]

$$\mathbf{E} = k^2\mathbf{\Pi} + \nabla\psi - i\omega\mu\nabla \times \mathbf{\Pi}^* \tag{15}$$

$$\mathbf{H} = (i\omega\epsilon + \sigma)\nabla \times \mathbf{\Pi} + k^2\mathbf{\Pi}^* + \nabla\psi^*, \tag{16}$$

$$k^2 = \mu\epsilon\omega^2 - i\mu\sigma\omega, \tag{17}$$

where μ, ϵ, σ are respectively permeability, dielectric constant, and conductivity in rationalized m.k.s. units. The electric and magnetic Hertzian vectors $\mathbf{\Pi}$ and $\mathbf{\Pi}^*$ are independent and satisfy

$$-\nabla \times \nabla \times \mathbf{\Pi} + k^2\mathbf{\Pi} + \nabla\psi = 0. \tag{18}$$

The scalars ψ and ψ^* are likewise independent. Usually they are placed equal to the divergence of the corresponding Hertz vector, but any solution of a homogeneous scalar wave equation may be employed.

Solutions of vector equations such as (14) and (18) can be constructed from solutions of the scalar wave equation.[3] Thus any rectangular component of a Hertzian vector satisfies (1). The differential operators may be expressed in terms of curvilinear coordinates and upon carrying out the operations indicated in (15) and (16) an electromagnetic field is obtained in a curvilinear system of coordinates. Such fields must

[2] J. A. Stratton, "Electromagnetic Theory," pp. 28–32, McGraw-Hill, 1941.

[3] J. A. Stratton, loc. cit., pp. 349–354, 392–399, 414–420.

eventually be superposed to satisfy boundary conditions on a prescribed surface. In place of a rectangular component it may be shown that the spherical function Π_R/R is also a solution of the scalar wave equation, where Π_R is a radial component. The field derived from a single component Π_R has no radial component of $\mathbf{H}$; that derived from Π_R^* has no radial component of $\mathbf{E}$. Unfortunately a vector field satisfying boundary conditions on a spheroid cannot be obtained in the same way from Hertz vectors which are everywhere normal to the surface of the spheroid. Such fields must be constructed by superposition of partial fields derived in the manner just described.

1.3 **Gegenbauer Functions.** The standard equation (7) is characterized by regular singularities at $z = \pm 1$ and an irregular point at infinity. The object of this investigation is to obtain independent solutions in the neighborhood of each singularity reduced to a form suitable for numerical calculations and to discuss their analytic continuation over the entire complex plane of z.

From (13) we note that if $f \to 0$, $\xi \to \infty$ in such a way that $f\xi \to R$, the spheroid reduces to a sphere of radius R. In this limit (7) is replaced by

$$z^2U'' + 2(a + 1)zU' + (c^2z^2 - b)U = 0, \tag{19}$$

an equation satisfied by half-order Bessel functions. For this reason the spheroidal functions will be represented in the neighborhood of the essential singularity at infinity by expansions in Bessel functions.

If, on the other hand, we place $c = 0$, (7) reduces to

$$(1 - z^2)U'' - 2(a + 1)zU' + bU = 0. \tag{20}$$

This is essentially the Gegenbauer equation, a hypergeometric equation with three regular singularities.[4] Since the spheroidal functions will be expressed in terms of the solutions of (20), we shall give a brief account of their properties.

The functions $T_n^a(z)$ and $\mathfrak{T}_n^a(z)$ are defined for all values of a, n, and z by

$$T_n^a(z) = (z^2 - 1)^{-a/2}P_{n+a}^a(z), \tag{21}$$

$$\mathfrak{T}_n^a(z) = (z^2 - 1)^{-a/2}Q_{n+a}^a(z), \tag{22}$$

where $P_{n+a}^a(z)$ and $Q_{n+a}^a(z)$ are the associated Legendre functions of the

[4] Whitaker and Watson, "Modern Analysis," p. 329, 4th ed., Cambridge University Press.

first and second kinds. Definitions of the Legendre functions differ in the literature with regard to phase, or the location of the cut in the z-plane. Accordingly, we shall adhere to the definitions given by Hobson.[5]

$$P_n^a(z) = \frac{e^{-n\pi i}(n+a)!}{4\pi \sin n\pi \, 2^n n!}(z^2-1)^{a/2} \times \int_C^{(z+,1+,z-,1-)} (t^2-1)^n (t-z)^{-n-a-1}\, dt \quad (23)$$

$$Q_n^a(z) = \frac{i\, e^{-n\pi i}(n+a)!}{4 \sin n\pi \, 2^n n!}(z^2-1)^{a/2} \times \int_C^{(-1+,1-)} (t^2-1)^n (t-z)^{-n-a-1}\, dt. \quad (24)$$

The functions T_n^a and $\mathfrak{T}_n^a$ differ slightly from those defined originally by Gegenbauer. They satisfy the equation

$$(z^2-1)w'' + 2(a+1)zw' - n(n+2a+1)w = 0. \quad (25)$$

But (25) is satisfied likewise by each of the following eight functions:

$$\begin{array}{llll} T_n^a, & T_{-n-2a-1}^a, & (z^2-1)^{-a}T_{n+2a}^{-a}, & (z^2-1)^{-a}T_{-n-1}^{-a}, \\ \mathfrak{T}_n^a, & \mathfrak{T}_{-n-2a-1}^a, & (z^2-1)^{-a}\mathfrak{T}_{n+2a}^{-a}, & (z^2-1)^{-a}\mathfrak{T}_{-n-1}^{-a}. \end{array}$$

Since the second order equation (25) admits only two independent solutions, certain linear relations must exist between the eight functions above. These follow directly from the theory of the hypergeometric function, and they are readily deduced from corresponding expressions for the Legendre functions given by Hobson.

$$T_n^a = T_{-n-2a-1}^a \quad (26)$$

$$T_{n+2a}^{-a} = T_{-n-1}^{-a} \quad (27)$$

$$\mathfrak{T}_a^n = e^{2a\pi i}\frac{(n+2a)!}{n!}(z^2-1)^{-a}\mathfrak{T}_{n+2a}^{-a} \quad (28)$$

$$\mathfrak{T}_{-n-2a-1}^a = e^{2a\pi i}\frac{(-n-1)!}{(-n-2a-1)!}(z^2-1)^{-a}\mathfrak{T}_{-n-1}^{-a} \quad (29)$$

$$T_n^a = \frac{e^{-a\pi i}}{\pi \cos (n+a)\pi}[\sin (n+2a)\pi\, \mathfrak{T}_n^a - \sin n\pi\, \mathfrak{T}_{-n-2a-1}^a] \quad (30)$$

[5] E. W. Hobson, "Spheroidal and Ellipsoidal Harmonics," pp. 188 and 195, Cambridge University Press, 1931.

$$T_{n+2a}^{-a} = \frac{e^{a\pi i}}{\pi \cos (n + a)\pi} [\sin n\pi\, \mathfrak{T}_{n+2a}^{-a} - \sin (n + 2a)\pi\, \mathfrak{T}_{-n-1}^{-a}] \tag{31}$$

$$T_n^a = \frac{e^{a\pi i}(n + 2a)!\sin (n + 2a)\pi}{\pi n! \cos (n + a)\pi} (z^2 - 1)^{-a}[\mathfrak{T}_{n+2a}^{-a} - \mathfrak{T}_{-n-1}^{-a}] \tag{32}$$

$$(z^2 - 1)^{-a} T_{n+2a}^{-a} = \frac{e^{-a\pi i} n! \sin n\pi}{\pi (n + 2a)! \cos (n + a)\pi} [\mathfrak{T}_n^a - \mathfrak{T}_{-n-2a-1}^a] \tag{33}$$

$$\frac{2}{\pi} e^{-a\pi i} \sin a\pi\, \mathfrak{T}_n^a = T_n^a - \frac{(n + 2a)!}{n!} (z^2 - 1)^{-a} T_{n+2a}^{-a} \tag{34}$$

$$\frac{2}{\pi} e^{-a\pi i} \sin a\pi\, \mathfrak{T}_{-n-2a-1}^a = T_n^a - \frac{(-n - 1)!}{(-n - 2a - 1)!} (z^2 - 1)^{-a} T_{n+2a}^{-a} \tag{35}$$

The factorial has been employed in place of the gamma function for non-integral as well as integral values of the argument, so that $a! = \Gamma(a + 1)$.

In addition to these formulas connecting the various solutions of (25) frequent use will be made of the recurrence relations

$$z\, T_n^a = \frac{1}{2n + 2a + 1} [(n + 1) T_{n+1}^a + (n + 2a)\, T_{n-1}^a] \tag{36}$$

$$(z^2 - 1) \frac{d}{dz} T_n^a = (n + 1)\, T_{n+1}^a - (n + 2a + 1) z\, T_n^a \tag{37}$$

which apply also to $\mathfrak{T}_n^a$.

From (21) and (22) and the known properties of Legendre functions the solutions of (25) can be expressed in terms of hypergeometric functions. Thus for $| 1 - z | < 2$,

$$T_n^a = \frac{1}{(-a)!} (z - 1)^{-a} F\left(- n - a, n + a + 1; 1 - a; \frac{1 - z}{2}\right), \tag{38}$$

while for $| z | > 1$

$$\mathfrak{T}_n^a = \frac{e^{a\pi i} 2^a \sqrt{\pi}\, (n + 2a)!}{(n + a + \frac{1}{2})!\, (2z)^{n+2a+1}} \times F\left(\frac{n + 2a + 2}{2}, \frac{n + 2a + 1}{2}, n + a + \frac{3}{2}, \frac{1}{z^2}\right) \tag{39}$$

where

$$F(a, b; c; z) = \frac{(c - 1)!}{(a - 1)!\, (b - 1)!} \sum_{s=0}^{\infty} \frac{(a + s - 1)!\, (b + s - 1)!}{s!\, (c + s - 1)!} z^s. \tag{40}$$

Through the transformations of the hypergeometric function and the relations (26)–(35) one may establish series expansions for the functions T and $\mathfrak{T}$ for the entire complex domain of z.

1.4 Gegenbauer Functions for Integer Values of a. Spheroidal wave functions associated with a complete spheroidal surface must be periodic in the equatorial angle φ. In this case a is a positive integer or zero and will be designated by the letter m. The relations between the functions defined in Sec. 1.3 then reduce as follows. For all values of n we have

$$T_n^m = T_{-n-2m-1}^m \tag{41}$$

$$T_{n+2m}^{-m} = T_{-n-1}^{-m} \tag{42}$$

$$T_n^m = \frac{(n+2m)!}{n!}(z^2-1)^{-m}T_{n+2m}^{-m} \tag{43}$$

$$\mathfrak{T}_n^m = \frac{(n+2m)!}{n!}(z^2-1)^{-m}\mathfrak{T}_{n+2m}^{-m} \tag{44}$$

$$\mathfrak{T}_{-n-2m-1}^m = \frac{(n+2m)!}{n!}(z^2-1)^{-m}\mathfrak{T}_{-n-1}^{-m} \tag{45}$$

$$T_n^m = \frac{\tan n\pi}{\pi}(\mathfrak{T}_n^m - \mathfrak{T}_{-n-2m-1}^m) \tag{46}$$

Numerical values of these functions in various regions of the z-plane may be obtained from the following hypergeometric series:

For $|z| > 1$,

$$\begin{aligned}\mathfrak{T}_n^{\pm m} &= \frac{(-2)^{\mp m}\sqrt{\pi}\,(n \pm 2m)!}{(n \pm m + \frac{1}{2})!\,(2z)^{n+1}} \\ &\quad \times (z^2-1)^{\mp m} F\left(\frac{n+2}{2}, \frac{n+1}{2}; n \pm m + \frac{3}{2}; \frac{1}{z^2}\right) \\ &= \frac{(-2)^{\pm m}\sqrt{\pi}\,(n \pm 2m)!}{(n \pm m + \frac{1}{2})!\,(2z)^{n \pm 2m+1}} \\ &\quad \times F\left(\frac{n \pm 2m + 2}{2}, \frac{n \pm 2m + 1}{2}; n \pm m + \frac{3}{2}; \frac{1}{z^2}\right)\end{aligned} \tag{47}$$

For $|1 - z| < 2$,

$$T_n^m = \frac{(n+2m)!}{2^m n!\, m!} F\left(-n, n+2m+1; 1+m; \frac{1-z}{2}\right), \tag{48}$$

$$
\begin{aligned}
\mathfrak{T}_n^m = \tfrac{1}{2} T_n^m \left[\ln \frac{z+1}{z-1} - \sigma(n) - \sigma(n+2m) \right] - \frac{\sin n\pi}{2\pi} \Bigg[(z-1)^{-m} \\
\times \sum_{r=0}^{m-1} \frac{(-1)^r (-n-m+r-1)!\,(n+m+r)!\,(m-r-1)!}{r!} \left(\frac{1-z}{2}\right)^r \\
+ 2^{-m} \sum_{r=0}^{\infty} \frac{(-n+r-1)!\;(n+2m+r)!}{r!\;(m+r)!}\, \sigma(r) \left(\frac{1-z}{2}\right)^r
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
+ (-1)^m \frac{(n+2m)!}{n!} (z+1)^{-m} \\
\times \sum_{r=0}^{\infty} \frac{(-n-m+r-1)!\;(n+m+r)!}{r!\;(r+m)!}\, \sigma(m+r) \left(\frac{1-z}{2}\right)^r \Bigg]
\end{aligned}
$$

wherein $n \geqq 0$ and

$$
\sigma(x) = \frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{x}. \tag{50}
$$

For $|z| < 1$ and $Im(z) \neq 0$,

$$
\begin{aligned}
\mathfrak{T}_n^m = \frac{i^{\mp(1+n)} \sqrt{\pi} \left(\dfrac{n+2m-1}{2}\right)!}{2^{1-m} \left(\dfrac{n}{2}\right)!} F\left(\frac{n+2m+1}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) \\
+ \frac{i^{\mp n} \sqrt{\pi} \left(\dfrac{n+2m}{2}\right)!\, 2^m}{\left(\dfrac{n-1}{2}\right)!} z\, F\left(\frac{n+2m+2}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right),
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
T_n^m = 2^m \cos \frac{n\pi}{2} \frac{\left(\dfrac{n+2m-1}{2}\right)!}{\sqrt{\pi} \left(\dfrac{n}{2}\right)!} F\left(\frac{n+2m+1}{2}, \frac{-n}{2}; \frac{1}{2}; z^2\right) \\
+ 2^{m+1} \sin \frac{n\pi}{2} \frac{\left(\dfrac{n+2m}{2}\right)!}{\sqrt{\pi} \left(\dfrac{n-1}{2}\right)!} z\, F\left(\frac{n+2m+2}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right)
\end{aligned}
\tag{52}
$$

Usually we shall have to deal with functions for which n as well as m is a positive integer. In this case some of the above series break off and hence are finite for all finite values of z. Thus when both n and m

are positive integers the various solutions fall into two classes: a set of integral functions, and a set characterized by singularities at $z = \pm 1$ as well as at infinity.

TABLE I

n and m positive integers

T_n^m	$(z^2-1)^{-m}T_{n+2m}^{-m}$	no singularities in the z-plane other than at $z = \infty$
$T_{-n-2m-1}^m$	$(z^2-1)^{-m}T_{-n-1}^{-m}$	
$\mathfrak{T}_{-n-2m-1}^m$	$(z^2-1)^{-m}\mathfrak{T}_{-n-1}^{-m}$	
$\mathfrak{T}_n^m$	$(z^2-1)^{-m}\mathfrak{T}_{n+2m}^{-m}$	singularities at $z = \pm 1$ and $z = \infty$

If n is a positive or negative integer $\mathfrak{T}_n^m$ becomes infinite when $n \leqq -2m - 1$ in virtue of the factor $(n + 2m)!$ in the numerator. It is finite if $n > -2m - 1$. Likewise T_n^m is zero when $-2m - 1 < n < 0$. It is finite for all other values of n.

1.5 **Gegenbauer Functions for $a = -\frac{1}{2}$.** Comparable in importance to the case $a = m$ is that in which $a = -\frac{1}{2}$. We have seen that this leads to the Mathieu functions. If we let $z = \cosh \psi$, Eq. (20) reduces for $a = -\frac{1}{2}$ to

$$\frac{d^2U}{d\psi^2} - bU = 0 \tag{53}$$

satisfied by hyperbolic functions. From the definitions of T_n^a and $\mathfrak{T}_n^a$ it can be shown that

$$\mathfrak{T}_n^{-\frac{1}{2}} = -\frac{i}{n}\sqrt{\frac{\pi}{2}}\,e^{-n\psi} \tag{54}$$

$$(z^2-1)^{\frac{1}{2}}\mathfrak{T}_{n-1}^{\frac{1}{2}} = i\sqrt{\frac{\pi}{2}}\,e^{-n\psi} \tag{55}$$

$$T_0^{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\,\psi \tag{56}$$

$$T_n^{-\frac{1}{2}} = \frac{1}{n}\sqrt{\frac{2}{\pi}}\sinh n\psi \tag{57}$$

$$(z^2-1)^{\frac{1}{2}}T_{n-1}^{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\cosh n\psi \tag{58}$$

for all values of n. The relations between the functions reduce to

$$T_n^{-\frac{1}{2}} = T_{-n}^{-\frac{1}{2}} \tag{59}$$

$$\mathfrak{T}_n^{-\frac{1}{2}} = -\frac{1}{n}(z^2-1)^{\frac{1}{2}}\mathfrak{T}_{n-1}^{\frac{1}{2}} \tag{60}$$

$$T_n^{-\frac{1}{2}} = -\frac{i}{\pi}(\mathfrak{T}_n^{-\frac{1}{2}} + \mathfrak{T}_{-n}^{-\frac{1}{2}}) \tag{61}$$

$$(z^2-1)^{\frac{1}{2}}T_{n-1}^{\frac{1}{2}} = \frac{in}{\pi}(\mathfrak{T}_n^{-\frac{1}{2}} - \mathfrak{T}_{-n}^{-\frac{1}{2}}) \tag{62}$$

$$\mathfrak{T}_n^{-\frac{1}{2}} = \frac{i\pi}{2}\left[T_n^{-\frac{1}{2}} - \frac{1}{n}(z^2-1)^{\frac{1}{2}}T_{n-1}^{\frac{1}{2}}\right] \tag{63}$$

These are all integral functions, having no singularities other than at $z = \infty$.

II. General Spheroidal Functions

2.1 **Definition.** In Sec. 1.3 it was noted that Eq. (7) reduces to (19) when $z \gg 1$ and $c \to 0$, and is then satisfied by

$$U = (cz)^{-a-\frac{1}{2}}Z_{l+a+\frac{1}{2}}(cz) \tag{64}$$

where $Z_p(cz)$ is any Bessel function of order p and argument cz. Likewise when $c \to 0$ Eq. (7) reduces to (20) and is satisfied by T_l^a or $\mathfrak{T}_l^a$ provided b has the characteristic value $l(l + 2a + 1)$. In general we shall express b in the form

$$b = l(l + 2a + 1) + \epsilon_l(a, c) \tag{65}$$

where ϵ_l is a function of the parameters l, a and c, and vanishes with c. A circle drawn about the origin as a center and passing through the singular points at $z = \pm 1$ divides the z-plane into two domains and limits the convergence of most series representations. We consider two systems of solutions. Those solutions of (7) that are valid in the neighborhood of the essential singularity at infinity we designate by U. They shall be represented in terms of the Bessel functions (64). Those solutions about the ordinary point $z = 0$ and converging at least within a circle of unit radius are to be designated by V, and will be represented by expansions in Gegenbauer functions. These representations are remarkable in that only one set of expansion coefficients need be determined. The problems of numerical computations and of determining the connections linking the solutions in one domain with those in another are thus vastly simplified.

2.2 *U*-Functions. The solutions of (7) valid when $z \gg 1$ will be defined by the expansion

$$U(a, c; z) = (cz)^{-a-\frac{1}{2}} \sum_n a_n Z_{n+\rho+a+\frac{1}{2}}(cz), \tag{66}$$

where n is an integer and $0 \lesseqgtr \rho < 1$. Upon introducing this series into (7) and applying the recurrence relations for Bessel functions it is found that the coefficients a_n satisfy the three term recursion formula

$$\begin{aligned}\frac{(n+\rho+1)(n+\rho+2)}{(2n+2\rho+2a+3)(2n+2\rho+2a+5)}\, a_{n+2} \\ + \frac{(n+\rho+2a-1)(n+\rho+2a)}{(2n+2\rho+2a-3)(2n+2\rho+2a-1)}\, a_{n-2} \\ + \left[\frac{b-(n+\rho)(n+\rho+2a+1)}{c^2}\right. \\ \left. - \frac{2(n+\rho)^2 + 2(n+\rho)(2a+1) + 2a - 1}{(2n+2\rho+2a-1)(2n+2\rho+2a+3)}\right] a_n = 0.\end{aligned} \tag{67}$$

When $c \to 0$ there remains in the limit only

$$[b - (n+\rho)(n+\rho+2a+1)]a_n = 0.$$

If then we choose

$$b = (l+\rho)(l+\rho+2a+1),$$

where l is either an even or an odd integer, it is apparent that

$$a_n \to 0, \quad \text{for} \quad n \gtrless l$$

and U reduces to a single term of order $l + \rho + a + \frac{1}{2}$. When c differs from zero the summation extends from $-\infty$ to $+\infty$. As the subscripts of a_n in (67) differ always by an even number, the even and odd series are distinct: the summation extends over even values of n when l is even, over odd values when l is odd.

In order that (66) shall be an analytic representation of a solution it is necessary that the series converge for $cz > 1$. The maximum absolute magnitude of a Bessel function is independent of its order when $cz \to \infty$. The coefficients a_n must therefore converge in both directions from a_l, the dominant term. This condition, in turn, restricts the separation constant b to a set of characteristic values b_l.

The quantity ρ plays a role analogous to the roots of an indicial equation in the power series solution of a differential equation. By proper

choice of ρ it is possible to terminate a certain number of the expansions at $n = 0$ and thus to restrict the summation to positive values of n. However, this is untrue in other cases and in general the summation must be extended from $-\infty$ to $+\infty$, over either even or odd values of n. An earlier paper[6] failed to take this into account and the conclusions drawn there must be modified accordingly.

If now we let

$$U(a, c; z) = (z^2 - 1)^{-a} X(a, c, z), \tag{68}$$

it is apparent that X must satisfy the equation

$$(z^2 - 1)X'' + 2(1 - a)zX' + (c^2z^2 - b - 2a)X = 0 \tag{69}$$

and X can be represented by expansions of the form

$$X(a, c; z) = (cz)^{a-\frac{1}{2}} \sum_{n'} a'_{n'} Z_{n'-a+\frac{1}{2}}(cz), \tag{70}$$

where the $a'_{n'}$ satisfy the recursion formula

$$\begin{aligned} &\frac{(n'+1)(n'+2)}{(2n'-2a+3)(2n'-2a+5)}\, a'_{n'+2} \\ &\qquad + \frac{(n'-2a-1)(n'-2a)}{(2n'-2a-3)(2n'-2a-1)}\, a'_{n'-2} \\ &\qquad + \left[\frac{b+2a-(n'-2a+1)n'}{c^2} \right. \\ &\qquad \left. - \frac{2n'^2+2n'(1-2a)-2a-1}{(2n'-2a+3)(2n'-2a-1)}\right] a'_{n'} = 0. \end{aligned} \tag{71}$$

Upon comparing (71) with the recursion formula (67) we observe that the two are identical if

$$\begin{aligned} n' &= n + \rho + 2a, \\ a'_{n'} &= \frac{(n+\rho)!}{(n+\rho+2a)!}\, Ca_n, \end{aligned} \tag{72}$$

where C is any constant. Since the Bessel functions of negative order satisfy the same relations as those of positive order, further solutions of (7) are obtained involving the same coefficients a_n.

We shall define a set of eight spheroidal functions as follows in terms of the Bessel functions of the first and second kinds. The normalization of the coefficients will be discussed later.

$$U_1(a, c; z) = (cz)^{-a-\frac{1}{2}} {\sum_n}' a_n J_{n+\rho+a+\frac{1}{2}}(cz) \tag{73}$$

[6] J. A. Stratton, Proc. Nat. Acad. Sci., *6*, 51–56, 1935 and *6*, 316–321, 1935.

$$U_2(a, c; z) = (cz)^{-a-\frac{1}{2}} \sum_n{}' a_n J_{-n-\rho-a-\frac{1}{2}}(cz) \tag{74}$$

$$U_3(a, c; z) = (cz)^{a-\frac{1}{2}}(z^2 - 1)^{-a} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} J_{n+\rho+a+\frac{1}{2}}(cz) \tag{75}$$

$$U_4(a, c; z) = (cz)^{a-\frac{1}{2}}(z^2 - 1)^{-a} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} J_{-n-\rho-a-\frac{1}{2}}(cz) \tag{76}$$

$$U_5(a, c; z) = (cz)^{-a-\frac{1}{2}} \sum_n{}' a_n N_{n+\rho+a+\frac{1}{2}}(cz) \tag{77}$$

$$U_6(a, c; z) = (cz)^{-a-\frac{1}{2}} \sum_n{}' a_n N_{-n-\rho-a-\frac{1}{2}}(cz) \tag{78}$$

$$U_7(a, c; z) = (cz)^{a-\frac{1}{2}}(z^2 - 1)^{-a} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} N_{n+\rho+a+\frac{1}{2}}(cz) \tag{79}$$

$$U_8(a, c; z) = (cz)^{a-\frac{1}{2}}(z^2 - 1)^{-a} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!} N_{-n-\rho-a-\frac{1}{2}}(cz). \tag{80}$$

The coefficients a_n of all these solutions satisfy the recursion formula (67), and summation is extended from negative infinity to positive infinity. The prime over the summation sign indicates that only even values of n are to be taken if the index l of the separation constant is even, only odd values if l is odd.

2.3 **V-Functions.** We construct next a set of solutions for the domain $|z| < 1$. When $c = 0$ Eq. (7) is satisfied by $T^a_{l+\rho}(z)$ provided that the characteristic value b_l belongs to the set $(l+\rho)(l+\rho+2a+1)$. If, therefore, c does not vanish, one is led to try solutions of the form

$$V(a, c; z) = \sum_n{}' d_n T^a_{n+\rho}(z), \tag{81}$$

where n is an odd or even integer and $0 \leqq \rho < 1$. The l^{th} term of the series is dominant when c is small. Upon substitution of V into (7) and applying the differential and recurrence relations of the Gegenbauer functions it is found that the coefficients d_n satisfy the recursion formula

$$\begin{aligned} &\frac{(n+\rho+2a+2)(n+\rho+2a+1)}{(2n+2\rho+2a+5)(2n+2\rho+2a+3)} d_{n+2} \\ &\qquad + \frac{(n+\rho)(n+\rho-1)}{(2n+2\rho+2a-1)(2n+2\rho+2a-3)} d_{n-2} \\ &\qquad + \left[\frac{2(n+\rho)^2 + 2(n+\rho)(2a+1) + 2a - 1}{(2n+2\rho+2a-1)(2n+2\rho+2a+3)}\right. \\ &\qquad + \left.\frac{(n+\rho)(n+\rho+2a+1) - b}{c^2}\right] d_n = 0. \end{aligned} \tag{82}$$

By comparison of (82) and (67) it is easily shown that, apart from a proportionality constant to be determined by the normalization,

$$d_n = i^n \frac{(n+\rho)!}{(n+\rho+2a)!} a_n. \tag{83}$$

The recursion formula (82) is unmodified when $n + \rho$ is replaced by $-n - \rho - 2a - 1$, so that two more solutions can be constructed from the functions $T^a_{-n-\rho-2a-1}$, $\mathfrak{T}^a_{-n-\rho-2a-1}$. Furthermore, as in the case of the U-functions, another four solutions are obtained by the transformation

$$V(a, c; z) = (z^2 - 1)^{-a} Y(a, c; z), \tag{84}$$

where $Y(a, c; z)$ satisfies (69). Thus eight formal solutions of (7) are established which we designate as follows:

$$V_1(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n \mathfrak{T}^{-a}_{-n-\rho-1}(z) \tag{85}$$

$$V_2(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n \mathfrak{T}^{-a}_{n+\rho+2a}(z) \tag{86}$$

$$V_3(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} \mathfrak{T}^a_{-n-\rho-2a-1}(z) \tag{87}$$

$$V_4(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} \mathfrak{T}^a_{n+\rho}(z) \tag{88}$$

$$V_5(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n T^{-a}_{-n-\rho-1}(z) \tag{89}$$

$$V_6(a, c; z) = (z^2 - 1)^{-a} \sum_n{}' a_n i^n T^{-a}_{n+\rho+2a}(z) \tag{90}$$

$$V_7(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} T^a_{-n-\rho-2a-1}(z) \tag{91}$$

$$V_8(a, c; z) = \sum_n{}' a_n i^n \frac{(n+\rho)!}{(n+\rho+2a)!} T^a_{n+\rho}(z) \tag{92}$$

Again the prime over the summation sign indicates that odd or even integer values of n are to be taken according as l is odd or even. Unless otherwise specified the summation is from negative to positive infinity.

For large values of $|n|$ the absolute magnitude of the ratio T^a_n/T^a_{n+2} or $\mathfrak{T}^a_n/\mathfrak{T}^a_{n+2}$ approaches unity at either $z = 0$ or $z = \pm 1$. Thus the convergence of the V-functions, as was that of the U-functions, is determined by the convergence of the sum $\sum_{n=-\infty}^{\infty}{}' a_n$.

2.4 Calculation of the Coefficients a_n and the Separation Constant. For sufficiently small values of c one may express the separation constant b and the coefficients a_n as power series in c. These series are then substituted in the recursion formula (67). Upon collecting terms and equating coefficients of like powers of c to zero one obtains in a single process both the characteristic values b and the expansion coefficients a_n. However, the convergence of these expressions is usually slow and they prove unsatisfactory for values of $c^2 > 1$. At present the most useful treatment of three term recursion formulas is based on a method of continued fractions.[7] We shall describe briefly the procedure.

To avoid cumbersome factors let us write the fundamental recursion formula (67) in the form

$$A_{n+2}a_{n+2} + B_n a_n + C_{n-2}a_{n-2} = 0 \tag{93}$$

where

$$\begin{aligned} A_n &= \frac{(n+\rho)(n+\rho-1)}{(2n+2\rho+2a-1)(2n+2\rho+2a+1)}, \\ B_n &= \frac{b-(n+\rho)(n+\rho+2a+1)}{c^2} \\ &\qquad - \frac{2(n+\rho)^2+2(n+\rho)(2a+1)+2a-1}{(2n+2\rho+2a-1)(2n+2\rho+2a+3)}, \\ C_n &= \frac{(n+\rho+2a+1)(n+\rho+2a+2)}{(2n+2\rho+2a+1)(2n+2\rho+2a+3)}. \end{aligned} \tag{94}$$

Evidently (93) can be developed as a continued fraction. Thus the ratio a_l/a_{l-2} is

$$\begin{aligned} \frac{a_l}{a_{l-2}} &= -\cfrac{C_{l-2}}{B_l + A_{l+2}\cfrac{a_{l+2}}{a_l}} \\ &= -\cfrac{C_{l-2}}{B_l - \cfrac{A_{l+2}C_l}{B_{l+2} - \ddots - \cfrac{A_n C_{n-2}}{B_n + A_{n+2}\cfrac{a_{n+2}}{a_n}}}}. \end{aligned} \tag{95}$$

[7] E. L. Ince, Phil. Mag., *6*, 547–558, 1928.

If the series is to converge, the value of a_l/a_{l-2} will be unaffected by the last term a_{n+2}/a_n provided n is chosen sufficiently large. In practice one may usually start with n ten or twelve integers larger than l, neglect a_{l+12}/a_{l+10}, and upon running up the fraction an expression for the initial ratio a_l/a_{l-2} is obtained in terms of b which is contained explicitly in B. Repetition of this process with a neighboring value of n will indicate whether or not a sufficient number of terms have been taken into account.

Next the ratio a_l/a_{l-2} is approached in the opposite direction. Eq. (93) can be developed in the form

$$\begin{aligned}\frac{a_{l-2}}{a_l} &= -\cfrac{A_l}{B_{l-2} + C_{l-4}\cfrac{a_{l-4}}{a_{l-2}}} \\ &= -\cfrac{A_l}{B_{l-2} - \cfrac{A_{l-2}C_{l-4}}{B_{l-4} - \ddots - \cfrac{A_{-n+2}C_{-n}}{B_{-n} + C_{-n-2}\cfrac{a_{-n-2}}{a_{-n}}}}}.\end{aligned} \tag{96}$$

Again the ratio a_{-n-2}/a_{-n} can be neglected for a sufficiently large value of n, and upon running up the fraction an expression is obtained for a_{l-2}/a_l in terms of b. The reciprocal of this ratio must equal the value of a_l/a_{l-2} obtained from (95). Thus (95) and (96), when properly related, constitute a transcendental equation in b whose roots are the characteristic values. When these values have been determined, the continued fractions provide an easy means of computing the coefficients a_n in terms of any arbitrary coefficient a_l.

2.5 **Termination of the Series.** In general the series expansions of the U- and V-functions extend from minus to plus infinity. However, by a proper choice of ρ they may be terminated in certain cases at $n = 0$ or $n = 1$ and extended over positive values of n only. Thus for the even series

$$\frac{a_{-2}}{a_0} = -\cfrac{A_0}{B_{-2} + C_{-4}\cfrac{a_{-4}}{a_{-2}}}, \tag{97}$$

with

$$A_0 = \frac{\rho(\rho - 1)}{(2\rho + 2a - 1)(2\rho + 2a + 1)}. \tag{98}$$

Then if a_0 is finite and $a \neq \pm\frac{1}{2}$, a_{-2} vanishes as $\rho \to 0$. Moreover, it follows from (96) that all the remaining coefficients a_n with negative n vanish in the same manner, while the ratio a_{n-2}/a_n remains finite. Since $a_{-2} = 0$, the coefficients with positive subscripts can all be expressed in terms of a_0. From (96) and (95) one obtains

$$\begin{aligned} \frac{a_2}{a_0} &= -\frac{B_0}{A_2}, \\ \frac{a_2}{a_0} &= -\frac{C_0}{B_2 + A_4\dfrac{a_4}{a_2}}. \end{aligned} \tag{99}$$

As in Sec. 2.4 the separation constant b is to be determined such that the two expressions for the ratio a_2/a_0 are equal. The treatment of the odd series is identical.

Since the functions $J_p(cz)$ and $N_p(cz)$ are finite for negative orders p, it is apparent that the functions U_1, U_2, U_5, and U_6 defined in Sec. 2.2 are summed only over positive values of n, starting at $n = 0$ or $n = 1$ as l is even or odd. The remaining functions U_3, U_4, U_7, U_8 contain terms involving negative values of n in virtue of the factor $(n + \rho)!/(n + \rho + 2a)!$. Thus if $a = m$, a positive integer, the limit of $a_n(n + \rho)!/(n + \rho + 2m)!$ as $\rho \to 0$ is finite when $n > -2m - 1$, but zero for all values equal to or less than this number. The series again break off on the negative side, but at $n = -2m$ and $n = -2m + 1$ for the even and odd cases respectively. When a is a negative integer the series also terminate. If $a = -m$, the even and odd series start at $n = 2m$ and $n = 2m + 1$ respectively, where m is a positive integer, and extend to infinity in the positive direction. Finally, when a is not an integer, all terms of the summation, both positive and negative, must be included.

Similar considerations hold for the V-functions. If $a = m$, the functions V_1, V_3, V_5, V_6, V_7, V_8 are summed over positive values of n only, starting at $n = 0$ or $n = 1$. Actually V_1 and V_3 are infinite in virtue of a factor $1/\rho$, but since this factor is common to each term of the series it can be removed. The summations for the functions V_2 and V_4 extend from negative to positive infinity, *even when a is a positive integer*. The ratio a_n/ρ for negative values of n was shown to be finite.

Since $\mathfrak{T}^{-m}_{n+\rho+2m}(z)$ for $n < 0$ and $\mathfrak{T}^{m}_{n+\rho}$ for $n < -2m$ contain a factor $1/\rho$, the products

$$a_n \mathfrak{T}^{-m}_{n+\rho+2m} \quad \text{and} \quad a_n \frac{(n+\rho)!}{(n+2m+\rho)!} T^m_{n+\rho}$$

are likewise finite.

2.6 **Mathieu Functions.** The special case $a = -\frac{1}{2}$ is of particular interest since it pertains to the Mathieu functions. Taking $\rho = 0$, one now obtains

$$A_n = C_n = \tfrac{1}{4},$$

$$B_n = \frac{b - n^2}{c^2} - \frac{1}{2},$$

and the recursion formula reduces to

$$a_{n+2} + a_{n-2} - \left[2 + \frac{4}{c^2}(n^2 - b)\right] a_n = 0. \tag{100}$$

Unlike the other cases, this relation is the same for both positive and negative values of n. Moreover, the Bessel and Gegenbauer functions are now either even or odd functions of the independent variable z. We shall construct U- and V-functions that satisfy one or the other of two conditions. For the first type we shall take

$$a_n = (-1)^n a_{-n}. \tag{101}$$

The series fails to break off, but in virtue of the symmetry of positive and negative terms it can be summed from $n = 0$ or $n = 1$ to $n = \infty$. In this case a_0 differs from zero. When the index l of the separation constant is even one obtains b from the relation

$$\frac{a_2}{a_0} = 1 - \frac{2b}{c^2} = -\frac{1}{4B_2 + a_4/a_2} \tag{102}$$

which can be expanded as a continuous fraction as in (95). Likewise when l is odd, the characteristic values of b are given by the roots of

$$\frac{a_3}{a_1} = 3 + \frac{4}{c^2}(1 - b) = -\frac{1}{4B_3 + a_5/a_3}. \tag{103}$$

For the second type we shall have

$$a_n = -(-1)^n a_{-n}. \tag{104}$$

Obviously $a_0 = 0$ and the characteristic values of b corresponding to even values of l are obtained from

$$\frac{a_4}{a_2} = 2 + \frac{4}{c^2}(4 - b) = -\frac{1}{4B_4 + a_6/a_4}, \tag{105}$$

while the roots of

$$\frac{a_3}{a_1} = 1 + \frac{4}{c^2}(1 - b) = -\frac{1}{4B_3 + a_5/a_3} \tag{106}$$

are the characteristic values of b associated with odd values of l. The relation of the functions constructed in this way to the Mathieu functions defined by other authors will be shown below.

The U- and V-functions themselves have certain properties that are peculiar to the special case $a = -\frac{1}{2}$, $\rho \to 0$. We now find $(n + \rho)!/(n + 2a + \rho)! = n$, $n + \rho + a + \frac{1}{2} = n$, and since for any Bessel function of integral order $Z_n = (-1)^n Z_{-n}$, there are two cases to be considered.

1. *Even Functions* $a_n = (-1)^n a_{-n}$. Then U_3, U_4, U_7, U_8 vanish, for the positive half of the series is equal but opposite in sign to the negative. The remaining functions U_1, U_2, U_5, U_6 are of the form

$$\sum_n{}' a_n Z_{\pm n}(cz).$$

Likewise V_7 and V_8 vanish, while the remaining V-functions can be put into the form

$$\sum_n{}' a_n i^n \cosh n\psi,$$

where $\psi = \cosh^{-1} z$, and are therefore proportional to each other. Thus when $\rho = 0$ we have only one independent V-function. A second solution in this case must be found by a limiting process. Mathieu functions constructed in terms of cosh $n\psi$ alone are even functions of ψ, and it is apparent that this has nothing to do with the odd or even character of l.

2. *Odd Functions*, $a_n = -(-1)^n a_{-n}$. The functions U_1, U_2, U_5, U_6 now vanish while U_3, U_4, U_7, U_8 are all of the form

$$\frac{\sqrt{z^2 - 1}}{cz} \sum_n{}' n a_n Z_{\pm n}.$$

Likewise V_5 and V_6 vanish and the remaining V-functions are all of the form

$$\sum_n{}' i^n a_n \sinh n\psi$$

and hence proportional to each other. They are odd functions of ψ, for even as well as odd values of l.

2.7 **Convergence.** The recursion formulas associated with equations of Mathieu type are of three terms. Consequently the convergence theory is more difficult than that of hypergeometric equations whose recursion formulas are always of two terms. In the present section we shall indicate the nature of the convergence and the conditions which appear to justify certain operations carried out in the remainder of this paper. The correctness of the results is supported also by extensive numerical calculations. A more thorough investigation is nevertheless desirable.

According to (93) the recursion formula relating the expansion coefficients can be written in the form

$$\frac{a_n}{a_{n-2}} = - \frac{C_{n-2}}{B_n + A_{n+2}\dfrac{a_{n+2}}{a_n}} \tag{107}$$

Let $r_n = a_n/a_{n-2}$. Then as $n \to \infty$,

$$r_n \to \frac{c^2}{2c^2 - 4b + 4n^2 - c^2 r_{n+2}}. \tag{108}$$

There are two possibilities. The separation constant b may be such that the denominator of (107) approaches zero with increasing n. The ratio r_n increases as n^2 and the series $\sum a_n$ diverges for all values of c. But b may also be chosen such that $r_{n+2} \to 0$. This was the procedure described in the foregoing sections and it is now apparent that for these characteristic values of b

$$r_n \to \frac{c^2}{4n^2}. \tag{109}$$

The series of coefficients a_n converges accordingly.

It is interesting to note the origin of these two sets of b, the one leading to divergent, the other to convergent series. The recursion formulas for the coefficients constitute a system of *difference* equations. A two-term difference equation is the counterpart of a first order dif-

ferential equation. One is free to specify the behavior of the solution in the neighborhood of $n = 0$, but the character of the solution as $n \to \infty$ is then fixed. A three-term difference equation corresponds on the other hand to a differential equation of second order, to which there are two independent solutions. Thus we may specify the value of the ratio r_n for small values of n, but this condition is no longer sufficient to determine the behavior of r_n at infinity. From the theory of difference equations it is easy to find asymptotic solutions when $n \to \infty$. The reader may verify that there are in fact two, the one becoming infinite, the other vanishing with increasing n. Of these two we choose in effect the latter, continue it analytically into the region of small n, and then select b such that the function r_n assumes the proper value at $n = 0$ or 1.

Consider now the U- and V-functions and examine the behavior of Bessel and Gegenbauer functions whose order is very much larger than the argument. Let $cz = p \operatorname{sech} \alpha \leqq p$. As $p \to \infty$, one must have $\alpha \to \infty$ if cz is to remain finite, or $\tanh \alpha \to 1$. Then according to Watson[8], when α is any fixed positive number and p is large and positive, one obtains the following asymptotic representation:

$$J_p(p \operatorname{sech} \alpha) \to \frac{e^{p(\tanh \alpha - \alpha)}}{\sqrt{2\pi p \tanh \alpha}}. \tag{110}$$

Since $\alpha \gg \tanh \alpha$, (110) vanishes exponentially with increasing p. On the other hand

$$N_p(p \operatorname{sech} \alpha) \to -\frac{e^{p(\alpha - \tanh \alpha)}}{\sqrt{\frac{1}{2}\pi p \tanh \alpha}}. \tag{111}$$

This function grows large without limit as $p \to \infty$, and the same is true of $J_{-p}(z)$ when p is not an integer. Nonetheless the corresponding U-functions converge in virtue of the rapid decrease in the coefficients a_n. Thus if $p = n + \rho + a + \frac{1}{2}$,

$$\frac{a_n N_p(cz)}{a_{n-2} N_{p-2}(cz)} \to \frac{c^2}{4n^2} e^{2(\alpha - \tanh \alpha)} \tag{112}$$

as $n \to \infty$. The asymptotic formulas (110) and (111) hold only when $p \gg cz \gg 1$. Hence the convergence of the U-functions has been demonstrated only for large values of the argument. Presumably the expansions (73)–(80) are convergent throughout the entire domain ex-

[8] G. N. Watson, "Theory of Bessel Functions," p. 243, Cambridge University Press, 1922.

terior to a circle of unit radius about the origin $z = 0$. From a practical point of view, however, these series are useful in most cases only when z is large. The functions $N_p(cz)$ grow large when the order p is equal or greater than the argument. If $z \simeq 1$, the first few terms increase rapidly and many terms must be employed before the rate of decrease in the coefficients a_n is sufficient to establish convergence.

When $|z| < 1$, all series containing terms in $N_p(cz)$ or $J_{-p}(cz)$ diverge. This is easily shown when $cz \ll 1$. For then

$$\frac{a_n N_p(cz)}{a_{n-2} N_{p-2}(cz)} \simeq \frac{1}{z^2}. \tag{113}$$

Exceptions occur when $a = m$ or $a = -\frac{1}{2}$. It has been shown that in these cases certain series can be terminated. All expansions in terms of Bessel functions of the first kind summed over positive orders only converge over the entire z-plane, including the points $z = \pm 1$.

The convergence of the V-functions is determined by the behavior of Gegenbauer functions of larger order. The required asymptotic representations are given by Hobson.[9] If the coefficients a_n diminish according to (109), it can be shown that the expansions (85–92) are convergent both in the interior of a circle of unit radius about the origin and in the region exterior to that circle. At the points $z = \pm 1$ the value of the functions may become infinite owing to a logarithmic factor or a factor of the form $(z^2 - 1)^{-a}$.

2.8 **Expansions of U- and V-Functions in Power Series.** There remains the fundamental problem of establishing analytic connections between the 16 functions defined in the preceding sections. This will be accomplished by first representing the various functions in terms of power series; coefficients can then be compared. In the domain $|z| > 1$ we shall employ power series representations of Bessel and Gegenbauer functions that are absolutely convergent. Terms may be grouped and the order changed arbitrarily. From the condition that the coefficients a_n form a convergent series it can be shown that the power series expansions of U- and V-functions obtained in this way likewise converge. This in no way contradicts the fact that in general only asymptotic representations can be found in the neighborhood of an essential singularity. For these power series expansions are actually Laurent series valid within an annular region bounded internally by the circle $|z| = 1$ and

[9] Hobson, loc. cit., pp. 302–313.

externally by a circle of very large radius excluding the point at infinity.

Within this annular region we find for the first four U-functions:

$$U_1 = 2^{-a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{\rho} \sum_n{}' a_n \left(\frac{cz}{2}\right)^{n} \times \sum_{k=0}^{\infty} \frac{1}{k!(n+\rho+a+k+\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{114}$$

$$U_2 = 2^{-a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{-(2a+\rho+1)} \sum_n{}' a_n \left(\frac{cz}{2}\right)^{-n} \times \sum_{k=0}^{\infty} \frac{1}{k!(-n-\rho-a+k-\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{115}$$

$$U_3 = (z^2-1)^{-a}2^{a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{2a+\rho} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!}\left(\frac{cz}{2}\right)^{n} \times \sum_{k=0}^{\infty} \frac{1}{k!(n+\rho+a+k+\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{116}$$

$$U_4 = (z^2-1)^{-a}2^{a-\frac{1}{2}}\left(\frac{cz}{2}\right)^{-\rho-1} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!}\left(\frac{cz}{2}\right)^{-n} \times \sum_{k=0}^{\infty} \frac{1}{k!(-n-\rho-a+k-\frac{1}{2})!}\left(\frac{icz}{2}\right)^{2k} \tag{117}$$

When $|z| > 1$ the $\mathfrak{T}$-functions are represented by the series (39). In this domain the first four V-functions assume the form

$$V_1 = (-\tfrac{1}{2})^{-a}\sqrt{\pi}(2z)^{\rho} \sum_n{}' a_n \frac{(-n-\rho-2a-1)!}{(-n-\rho-1)!}(i2z)^{n} \times \sum_{r=0}^{\infty} \frac{(2r-n-\rho-1)!}{r!(r-a-n-\rho-\frac{1}{2})!}(2z)^{-2r} \tag{118}$$

$$V_2 = (-\tfrac{1}{2})^{-a}\sqrt{\pi}(2z)^{-(2a+\rho+1)} \sum_n{}' a_n \frac{(n+\rho)!}{(n+\rho+2a)!}(-i2z)^{-n} \times \sum_{r=0}^{\infty} \frac{(2r+n+\rho+2a)!}{r!(r+n+\rho+a+\frac{1}{2})!}(2z)^{-2r} \tag{119}$$

$$V_3 = (-\tfrac{1}{2})^{a}\sqrt{\pi}(z^2-1)^{-a}(2z)^{2a+\rho}\frac{\sin(2a+\rho)\pi}{\sin\rho\pi} \sum_n{}' a_n(i2z)^{n} \times \sum_{r=0}^{\infty} \frac{(2r-n-\rho-2a-1)!}{r!(r-n-\rho-a-\frac{1}{2})!}(2z)^{-2r} \tag{120}$$

$$V_4 = (-\tfrac{1}{2})^a \sqrt{\pi}\,(z^2 - 1)^{-a}(2z)^{-\rho-1} \sum_n{}' a_n(-i2z)^{-n} \times \sum_{r=0}^{\infty} \frac{(2r + n + \rho)!}{r!\,(r + n + \rho + a + \frac{1}{2})!}\,(2z)^{-2r} \tag{121}$$

Let s be an integer which differs from n always by an even integer. The eight solutions can be arranged as follows:

$$U_1 = 2^{-a-\frac{1}{2}} \left(\frac{cz}{2}\right)^{\rho} \sum_s{}' \left(\frac{cz}{2}\right)^{s} \times \sum_{n \lesseqgtr s}{}' \frac{a_n i^{s-n}}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2} + \rho + a\right)!} \tag{122}$$

$$U_2 = 2^{-a-\frac{1}{2}} \left(\frac{cz}{2}\right)^{-2a-1-\rho} \sum_s{}' \left(\frac{cz}{2}\right)^{-s} \times \sum_{n \geqq s}{}' \frac{a_n i^{n-s}}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2} - \rho - a\right)!} \tag{123}$$

$$U_3 = 2^{a-\frac{1}{2}}(z^2 - 1)^{-a} \left(\frac{cz}{2}\right)^{2a+\rho} \sum_s{}' \left(\frac{cz}{2}\right)^{s} \times \sum_{n \lesseqgtr s}{}' \frac{a_n i^{s-n}(n+\rho)!}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2} + a + \rho\right)!\,(n + \rho + 2a)!} \tag{124}$$

$$U_4 = 2^{a-\frac{1}{2}}(z^2 - 1)^{-a} \left(\frac{cz}{2}\right)^{-\rho-1} \sum_s{}' \left(\frac{cz}{2}\right)^{-s} \times \sum_{n \geqq s}{}' \frac{a_n i^{n-s}(n+\rho)!}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2} - a - \rho\right)!\,(n + \rho + 2a)!} \tag{125}$$

$$V_1 = (-\tfrac{1}{2})^{-a}\sqrt{\pi}\,(2z)^{\rho} \sum_s{}' (2z)^{s}(-s - \rho - 1)! \times \sum_{n \geqq s}{}' \frac{a_n i^{n}(-n - \rho - 2a - 1)!}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2} - a - \rho\right)!\,(-n - \rho - 1)!} \tag{126}$$

$$V_2 = (-\tfrac{1}{2})^{-a}\sqrt{\pi}\,(2z)^{-2a-1-\rho} \sum_s{}' (2z)^{-s}(s + 2a + \rho)! \times \sum_{n \lesseqgtr s}{}' \frac{a_n i^{n}(n+\rho)!}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2} + a + \rho\right)!\,(n + 2a + \rho)!} \tag{127}$$

$$V_3 = (-\tfrac{1}{2})^a \sqrt{\pi}\,(z^2 - 1)^{-a}(2z)^{2a+\rho}\,\frac{\sin(2a+\rho)\pi}{\sin\rho\pi}$$
$$\times \sum_s{}' (2z)^s(-s - 2a - \rho - 1)! \tag{128}$$
$$\times \sum_{n \geq s}{}' \frac{a_n i^n}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2} - a - \rho\right)!}$$

$$V_4 = (-\tfrac{1}{2})^a \sqrt{\pi}\,(z^2 - 1)^{-a}(2z)^{-\rho-1} \sum_s{}' (2z)^{-s}(s+\rho)!$$
$$\times \sum_{n \lesseqgtr s}{}' \frac{a_n i^n}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2} + a + \rho\right)!} \tag{129}$$

2.9 **Relations between the Functions.** Since the eight functions defined in Sec. 2.8 satisfy one and the same equation, there can be at most two that are linearly independent within a common region of convergence. One observes that, so far as powers of z are concerned, each U-function is identical with the V-function of the same subscript, and we conclude that the two are proportional. Thus

$$\begin{aligned} U_1 &= K_1V_1, & U_2 &= K_2V_2, \\ U_3 &= K_3V_3, & U_4 &= K_4V_4. \end{aligned} \tag{130}$$

The proportionality constant K is called the joining factor. Upon equating the power series expansions of U and V the factor K is given as the ratio of coefficients of corresponding powers of z.

To obtain further relations between the various solutions we make use of the formula defining the Bessel function of the second kind,

$$N_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} \tag{131}$$

together with the functional relations between the Gegenbauer functions given in Sec. 1.3. Through these it is easy to establish the following connections.

Relations among the U-functions

$$U_1 = \frac{U_5 \sin(a+\rho)\pi + (-1)^n U_6}{\cos(a+\rho)\pi} \tag{132}$$

$$U_2 = -\frac{U_6 \sin(a+\rho)\pi + (-1)^n U_5}{\cos(a+\rho)\pi} \tag{133}$$

$$U_3 = \frac{U_7 \sin (a + \rho)\pi + (-1)^n U_8}{\cos (a + \rho)\pi} \tag{134}$$

$$U_4 = -\frac{U_8 \sin (a + \rho)\pi + (-1)^n U_7}{\cos (a + \rho)\pi} \tag{135}$$

$$U_5 = -\frac{U_1 \sin (a + \rho)\pi + (-1)^n U_2}{\cos (a + \rho)\pi} \tag{136}$$

$$U_6 = \frac{U_2 \sin (a + \rho)\pi + (-1)^n U_1}{\cos (a + \rho)\pi} \tag{137}$$

$$U_7 = -\frac{U_3 \sin (a + \rho)\pi + (-1)^n U_4}{\cos (a + \rho)\pi} \tag{138}$$

$$U_8 = \frac{U_4 \sin (a + \rho)\pi + (-1)^n U_3}{\cos (a + \rho)\pi} \tag{139}$$

Relations among the V-functions

$$V_1 = (-1)^{-2a} \frac{\sin \rho\pi}{\sin (2a + \rho)\pi} V_3 \tag{140}$$

$$V_2 = (-1)^{-2a} V_4 \tag{141}$$

$$V_5 = V_6 \tag{142}$$

$$V_7 = V_8 \tag{143}$$

$$(-1)^a \frac{2}{\pi} \sin a\pi V_1 = \frac{\sin \rho\pi}{\sin (2a + \rho)\pi} V_7 - V_5 \tag{144}$$

$$(-1)^a \frac{2}{\pi} \sin a\pi V_2 = V_7 - V_5 \tag{145}$$

$$(-1)^a \pi \cos (a + \rho)\pi V_5 = \sin \rho\pi (V_4 - V_3) \tag{146}$$

$$(-1)^a \pi \cos (a + \rho)\pi V_7 = \sin (2a + \rho)\pi (V_2 - V_1) \tag{147}$$

Relations between U- and V-functions

The power series expansions of the functions U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4 converge within the domain $1 < |z| < \infty$ when b has a characteristic value ensuring the absolute convergence of the coefficients a_n. Since U- and V-functions of equal subscript have power series of identical form, they must be proportional to each other and the proportionality factor $K_j = U_j/V_j$, ($j = 1, 2, 3, 4$), must equal the ratio of the coefficients of z^s in the corresponding series. Thus there are an infinite number of expressions for the joining factors, for s is any positive or negative integer differing from n or l by an even integer.

$$K_1 = \frac{(-1)^{a+s/2} c^{s+\rho}}{\sqrt{2\pi}\, 2^{2(s+a+\rho)}(-s-\rho-1)!} \times \frac{\displaystyle\sum_{n \leqq s}{}' \frac{i^{-n} a_n}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2}+a+\rho\right)!}}{\displaystyle\sum_{n \geqq s}{}' \frac{i^{n}(-n-2a-1-\rho)!\,a_n}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2}-a-\rho\right)!\,(-n-1-\rho)!}} \tag{148}$$

$$K_2 = \frac{(-1)^{a-s/2}\, 2^{2(s+a+1+\rho)}}{\sqrt{2\pi}\, c^{s+2a+1+\rho}(s+2a+\rho)!} \times \frac{\displaystyle\sum_{n \geqq s}{}' \frac{i^{n} a_n}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2}-a-\rho\right)!}}{\displaystyle\sum_{n \leqq s}{}' \frac{i^{n}(n+\rho)!\,a_n}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2}+a+\rho\right)!\,(n+2a+\rho)!}} \tag{149}$$

$$K_3 = \frac{(-1)^{-a+s/2} c^{s+2a+\rho}}{\sqrt{2\pi}\, 2^{2(s+a+\rho)}(-s-2a-1-\rho)!} \times \frac{\displaystyle\sum_{n \leqq s}{}' \frac{i^{-n}(-n-2a-1-\rho)!\,a_n}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2}+a+\rho\right)!\,(-n-1-\rho)!}}{\displaystyle\sum_{n \geqq s}{}' \frac{i^{n} a_n}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2}-a-\rho\right)!}} \tag{150}$$

$$K_4 = \frac{(-1)^{-a-s/2}\, 2^{2(s+a+1+\rho)}}{\sqrt{2\pi}\, c^{s+1+\rho}(s+\rho)!} \times \frac{\displaystyle\sum_{n \geqq s}{}' \frac{i^{n}(n+\rho)!\,a_n}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n+1}{2}-a-\rho\right)!\,(n+2a+\rho)!}}{\displaystyle\sum_{n \leqq s}{}' \frac{i^{n} a_n}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n+1}{2}+a+\rho\right)!}}. \tag{151}$$

Every value of s must, of course, lead to one and the same value of K_j.

2.10 **Summary of Relations when $a = m$.** When $a = m$, a positive integer, and $\rho \to 0$, the 16 U- and V-functions fall into two groups of independent solutions. The following table is so arranged that all functions in a group or column are identical.

TABLE II

Group 1	*Group 2*
$-\rho V_1$	V_2
$-\rho V_3$	V_4
V_5	U_2/K_2
V_6	U_4/K_4
V_7	$(-1)^{n+m+1}U_5/K_2$
V_8	$(-1)^{n+m+1}U_7/K_4$
$-\rho U_1/K_1$	
$-\rho U_3/K_3$	
$(-1)^{n+m+1}\rho U_6/K_1$	
$(-1)^{n+m+1}\rho U_8/K_3$	

By proper choice of s the joining factors reduce to the following. For l and n even:

$$K_1 = -\rho \frac{2^{-m-\frac{1}{2}} a_0}{(m+\frac{1}{2})!} [V_8(0)]^{-1} \tag{152}$$

$$K_2 = -\frac{2^{-m+\frac{3}{2}}}{c(m-\frac{3}{2})!\,(\rho - 2m)!\,a_{-2m}} V_8(0) \tag{153}$$

$$K_3 = \rho \frac{2^{m-\frac{1}{2}}(m-\frac{3}{2})!\,(\rho-2m)!\,a_{-2m}}{\pi} [V_8(0)]^{-1} \tag{154}$$

$$K_4 = \frac{2^{m+\frac{3}{2}}(m+\frac{1}{2})!}{\pi c a_0} V_8(0) \tag{155}$$

where

$$V_8(0) = V_8(a, c; z)\,|_{z=0} = \frac{1}{2^m\sqrt{\pi}} \sum_{n=0}^{\infty}{}' a_n \frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n}{2}+m\right)!}. \tag{156}$$

For l and n odd:

$$K_1 = -\rho \frac{2^{-m-\frac{3}{2}} c a_1}{(m+\frac{3}{2})!} [V_8'(0)]^{-1} \tag{157}$$

$$K_2 = \frac{2^{-m+\frac{1}{2}}}{c^2(m-\frac{5}{2})!\,(\rho+1-2m)!\,a_{1-2m}} V_8'(0) \tag{158}$$

$$K_3 = -\rho \frac{2^{m-\frac{1}{2}}c(m-\frac{5}{2})!\,(\rho+1-2m)!\,a_{1-2m}}{\pi} [V_8'(0)]^{-1} \tag{159}$$

$$K_4 = \frac{2^{m+\frac{1}{2}}(m+\frac{3}{2})!}{\pi c^2 a_1} V_8'(0), \tag{160}$$

where

$$V_8'(0) = \frac{d}{dz} V_8(a, c; z)\bigg|_{z=0} = \frac{i}{2^{m-1}\sqrt{\pi}} \sum_{n=1}' a_n \frac{\left(\frac{n}{2}\right)!}{\left(\frac{n-1}{2}+m\right)!} \tag{161}$$

The coefficients a_{-2m} and a_{1-2m} vanish as $\rho \to 0$ and $m > 0$, but the products $(\rho - 2m)!a_{-2m}$ and $(\rho + 1 - 2m)!a_{1-2m}$ are finite.

For both even and odd values of n

$$K_1K_4 = K_2K_3 = -\frac{2\rho}{\pi c}.$$

2.11 **Summary of Relations when** $a = -\frac{1}{2}$. When $a = -\frac{1}{2}$ and $\rho = 0$ the joining factors are reduced to the following by proper choice of s. These formulas apply both to the even and the odd Mathieu functions.

For l and n even:

$$K_1 = -i\sqrt{\frac{2}{\pi}}\, a_0 \left[a_0 + 2\sum_{n=2}^{\infty}{}' a_n\right]^{-1} \tag{163}$$

$$K_2 = -i\sqrt{\frac{2}{\pi}}\, a_0 \left[a_0 + 2\sum_{n=2}^{\infty}{}' a_{-n}\right]^{-1} \tag{164}$$

$$K_3 = i\frac{c}{8}\sqrt{\frac{2}{\pi}}\,(a_2 - a_{-2}) \left[\sum_{n=2}^{\infty} n a_n\right]^{-1} \tag{165}$$

$$K_4 = -i\frac{c}{8}\sqrt{\frac{2}{\pi}}\,(a_2 - a_{-2}) \left[\sum_{n=2}^{\infty} n a_{-n}\right]^{-1} \tag{166}$$

For l and n odd:

$$K_1 = \frac{c}{2\sqrt{2\pi}}(a_{-1} - a_1) \left[\sum_{n=1}^{\infty} n a_n\right]^{-1} \tag{167}$$

$$K_2 = \frac{c}{2\sqrt{2\pi}} (a_{-1} - a_1) \left[\sum_{n=1}^{\infty} na_{-n} \right]^{-1} \tag{168}$$

$$K_3 = -\frac{1}{2\sqrt{2\pi}} (a_{-1} + a_1) \left[\sum_{n=1}^{\infty} a_n \right]^{-1} \tag{169}$$

$$K_4 = \frac{1}{2\sqrt{2\pi}} (a_{-1} + a_1) \left[\sum_{n=1}^{\infty} a_{-n} \right]^{-1} \tag{170}$$

Even Functions, $a_n = (-1)^n a_{-n}$. The functions U_3, U_4, U_7, U_8, V_7, V_8 vanish as $\rho \to 0$, as do also the joining factors K_3 and K_4, and

$$K_1 = (-1)^n K_2. \tag{171}$$

The remaining functions again fall into two independent groups.

TABLE III

Group 1	*Group* 2
V_1	U_5
V_2	$(-1)^n U_6$
V_3	
$-V_4$	
$i\frac{\pi}{2} V_5$	
$i\frac{\pi}{2} V_6$	
U_1/K_1	
U_2/K_2	

As before, all functions occurring in one group or column are identical and represent one independent solution. The Gegenbauer functions for $a = -\frac{1}{2}$ can be expressed in terms of exponential and hyperbolic functions; by writing $z = \cosh \psi$ and doubling up the series one obtains:

l and n even,

$$V_1 = i\sqrt{\frac{\pi}{2}} \left[a_0 + 2 \sum_{n=2}^{\infty}{}' a_n i^n \cosh n\psi \right] \tag{172}$$

$$U_1 = a_0 J_0(cz) + 2 \sum_{n=2}^{\infty}{}' a_n J_n(cz) \tag{173}$$

$$U_5 = a_0 N_0(cz) + 2 \sum_{n=2}^{\infty}{}' a_n N_n(cz) \tag{174}$$

l and n odd,

$$V_1 = i\sqrt{2\pi} \sum_{n=1}^{\infty}{}' a_n i^n \cosh n\psi \tag{175}$$

$$U_1 = 2 \sum_{n=1}^{\infty}{}' a_n J_n(cz) \tag{176}$$

$$U_5 = 2 \sum_{n=1}^{\infty}{}' a_n N_n(cz). \tag{177}$$

V_1 and U_1 are Mathieu functions of the first kind and these representations converge over the entire z-plane. U_5 is a Mathieu function of the second kind, but the convergence of (174) and (177) is restricted to the region $|z| > 1$. A second solution of Mathieu's equation valid in the neighborhood of $z = 1$ has been discussed by Ince[10], but he fails to give the analytic connection with a corresponding solution valid when $z \to \infty$. In the next section we shall obtain another form of the second solution and the desired connections through application of a limiting process to the vanishing functions V_7 and V_8.

Odd Functions, $a_n = -(-1)^n a_{-n}$. In this case U_1, U_2, U_5, U_6, V_5, V_6, and the joining factors K_1 and K_2 vanish as $\rho \to 0$. Also

$$K_3 = (-1)^n K_4. \tag{178}$$

The remaining functions fall into two independent groups as in Table IV, the functions in any one column being all identical.

TABLE IV

Group 1	*Group* 2
V_1	U_7
$-V_2$	$(-1)^n U_8$
V_3	
V_4	
$i\frac{\pi}{2} V_7$	
$i\frac{\pi}{2} V_8$	
U_3/K_3	
U_4/K_4	

[10] E. L. Ince, Proc. Edinburgh Math. Soc., pp. 2–13, 1914–1915.

Upon doubling up the series so that the summation extends only over positive values of n one obtains

$$V_3 = i\sqrt{2\pi} \sum_{n=1,2}^{\infty}{}' a_n i^n \sinh n\psi, \tag{179}$$

$$U_3 = 2\frac{\sqrt{z^2-1}}{cz} \sum_{n=1,2}^{\infty}{}' a_n n J_n(cz), \tag{180}$$

$$U_7 = 2\frac{\sqrt{z^2-1}}{cz} \sum_{n=1,2}^{\infty}{}' a_n n N_n(cz). \tag{181}$$

When l is even these series start from $n = 2$, and from $n = 1$ when l is odd. As in the case of the even functions a second solution valid when $|z| \sim 1$ is missing, but will now be found by a limiting operation applied to the vanishing functions V_5 and V_6.

2.12 Second Solutions of the Mathieu Equation. The object of this section is to find a second solution of Mathieu's equation in the domain $|z| \sim 1$ and to join it analytically with representations of the same function valid when $|z| \gg 1$. In Sec. 2.11 it was shown that V_7 and V_8 for the even functions, and V_5 and V_6 for the odd functions, vanish as $\rho \to 0$. The zero is of the first order in ρ and hence can be removed if one divides by ρ before passing to the limit. Since ρ is an arbitrary constant, the functions obtained in this manner are still solutions of the differential equation.

Even Functions, $a_n = (-1)^n a_{-n}$. Formally

$$V_7 = V_8 = \sqrt{\frac{2}{\pi}} \sum_n{}' a_n i^n \sinh (n + \rho)\psi, \tag{182}$$

where the a_n are also functions of ρ. They vanish as $\rho \to 0$, since $a_n i^n \sinh n\psi = -a_{-n} i^{-n} \sinh(-n\psi)$. We now divide V_7 and V_8 by ρ, differentiate numerator and denominator with respect to ρ and pass to the limit $\rho = 0$.

$$\lim_{\rho\to 0} \frac{V_7}{\rho} = \lim_{\rho\to 0} \frac{V_8}{\rho} = \sqrt{\frac{2}{\pi}} \sum_n{}' i^n [a_n' \sinh n\psi + a_n \psi \cosh n\psi], \tag{183}$$

where

$$a_n' = \frac{d}{d\rho} a_n \bigg|_{\rho=0}. \tag{184}$$

In virtue of the relation

$$a_n' = -(-1)^n a_{-n}', \tag{185}$$

which will be established in the following section, the function defined by (183) is finite and does not vanish. It differs from all other V-functions in that it is an odd function of ψ, and it must, therefore, be a second solution of the Mathieu equation. It is this solution that was found by Ince[10] in another manner.

To find a relation between U- and V-functions of the second kind we make use of (136) for the case $a = -\frac{1}{2}$.

$$U_5 = \frac{U_1 \cos \rho\pi - (-1)^n U_2}{\sin \rho\pi}. \tag{186}$$

Upon replacing U_1 and U_2 by their equivalents in terms of V-functions and allowing $\rho \to 0$, one obtains

$$U_5 = \frac{1}{\pi} \lim_{\rho \to 0} \frac{d}{d\rho} [K_1 V_1 - (-1)^n K_2 V_2]. \tag{187}$$

Let

$$K' = \left.\frac{dK}{d\rho}\right|_{\rho=0}, \qquad V' = \left.\frac{dV}{d\rho}\right|_{\rho=0}. \tag{188}$$

Since $K_1 = (-1)^n K_2$, $V_1 = V_2$, we have

$$U_5 = \frac{1}{\pi} \{[K_1' - (-1)^n K_2']V_1 + K_1[V_1' - V_2']\}. \tag{189}$$

From (85), (86), and (55), when $\rho \neq 0$,

$$V_1 = i\sqrt{\frac{\pi}{2}} \sum_n{}' a_n i^n e^{(n+\rho)\psi} \tag{190}$$

$$V_2 = i\sqrt{\frac{\pi}{2}} \sum_n{}' a_n i^n e^{-(n+\rho)\psi}. \tag{191}$$

Thus

$$\begin{aligned} V_1' - V_2' &= i\sqrt{2\pi} \sum_n{}' i^n [a_n' \sinh n\psi + a_n \psi \cosh n\psi] \\ &= i\pi \frac{V_7}{\rho} = i\pi \frac{V_8}{\rho}. \end{aligned} \tag{192}$$

It appears that the solution U_5 valid when z is large is in fact a linear combination of solutions of the first and second kinds established for the region $|z| \sim 1$.

We proceed now to calculate the derivatives of the joining factors K. In (148) and (149) put $a = -\frac{1}{2}$ and obtain

$$K_1 = \frac{i\sqrt{\dfrac{2}{\pi}}\left(\dfrac{c}{4}\right)^{s+\rho} \displaystyle\sum_{n \lesseqgtr s}' \frac{a_n i^{s-n}}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n}{2}+\rho\right)!}}{(-s-1-\rho)!\displaystyle\sum_{n \geqq s}' \frac{a_n i^n (n+\rho)}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n}{2}-\rho\right)!}} \tag{193}$$

$$K_2 = -\frac{i\sqrt{\dfrac{2}{\pi}}\left(\dfrac{c}{4}\right)^{-s-\rho} \displaystyle\sum_{n \geqq s}' \frac{a_n i^{n-s}}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n}{2}-\rho\right)!}}{(s-1+\rho)!\displaystyle\sum_{n \lesseqgtr s}' \frac{a_n i^n (n+\rho)}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n}{2}+\rho\right)!}}. \tag{194}$$

For even values of n let $s = 0$. After simplification we obtain

$$\begin{aligned} K_1' &= -K_2' \\ &= K_1 \ln\frac{c}{4} - i\sqrt{\frac{2}{\pi}}\left\{\frac{a_0 C - 2\displaystyle\sum_{n=2}^{\infty}{}' \frac{a_n}{n}}{a_0 + 2\displaystyle\sum_{n=2}^{\infty}{}' a_n} \right. \\ &\qquad \left. - \frac{2a_0 \displaystyle\sum_{n=2}^{\infty}{}' \left[a_n' + a_n\left\{\sigma\left(\frac{n}{2}-1\right)+\frac{1}{n}\right\}\right]}{\left[a_0 + 2\displaystyle\sum_{n=2}^{\infty}{}' a_n\right]^2}\right\} \end{aligned} \tag{195}$$

where

$$C = 0.577216, \qquad \sigma(x) = \sum_{n=1}^{x} \frac{1}{n}.$$

For odd values of n let $s = 1$ in K_1 and $s = -1$ in K_2. Then

$$\begin{aligned} K_1' = K_2' = K_1 \ln\frac{c}{4} + \frac{c}{\sqrt{2\pi}} &\left\{\frac{-a_1(C+\frac{1}{2}) + 2\displaystyle\sum_{n=3}^{\infty}{}' \frac{a_n}{n^2-1}}{\displaystyle\sum_{n=1}^{\infty}{}' a_n n} \right. \\ &\left. + a_1 \sum_{n=1}^{\infty}{}' \frac{\left[a_n' n + a_n\left\{1 + n\sigma\left(\frac{n-1}{2}\right)\right\}\right]}{\left[\displaystyle\sum_{n=1}^{\infty}{}' a_n n\right]^2}\right\}. \end{aligned} \tag{196}$$

In general

$$K_1' = -(-1)^n K_2', \tag{197}$$

$$U_5 = \frac{2}{\pi} K_1' V_1 + iK_1 \frac{V_8}{\rho}. \tag{198}$$

Odd Functions, $a_n = -(-1)^n a_{-n}$. Formally

$$V_5 = V_6 = \sqrt{\frac{2}{\pi}} \sum_n{}' a_n i^n \cosh (n + \rho)\psi. \tag{199}$$

They vanish as $\rho \to 0$, since $a_n i^n \cosh n\psi = -a_{-n} i^{-n} \cosh (-n\psi)$. Upon division by ρ and passing to the limit $\rho \to 0$ one obtains

$$\lim_{\rho \to 0} \frac{V_5}{\rho} = \lim_{\rho \to 0} \frac{V_6}{\rho} = \sqrt{\frac{2}{\pi}} \sum_n{}' i^n [a_n' \cosh n\psi + a_n \psi \sinh n\psi]. \tag{200}$$

This is an even function of ψ, for

$$a_n' = (-1)^n a_{-n}', \tag{201}$$

as will be shown below.

To establish a relation between U- and V-functions use is made of (138).

$$U_7 = \frac{U_3 \cos \rho\pi - (-1)^n U_4}{\sin \rho\pi}. \tag{202}$$

Replacing U_3 and U_4 by their equivalent V-functions and letting $\rho \to 0$, we have

$$U_7 = \frac{1}{\pi} \lim_{\rho \to 0} \frac{d}{d\rho} [K_3 V_3 - (-1)^n K_4 V_4], \tag{203}$$

or

$$U_7 = \frac{1}{\pi} \{[K_3' - (-1)^n K_4'] V_3 + K_3 [V_3' - V_4']\}. \tag{204}$$

For arbitrary values of ρ

$$V_3 = i \sqrt{\frac{\pi}{2}} \sum_n{}' a_n i^n e^{(n+\rho)\psi}, \tag{205}$$

$$V_4 = -i \sqrt{\frac{\pi}{2}} \sum_n{}' a_n i^n e^{-(n+\rho)\psi}. \tag{206}$$

Thus

$$V_3' - V_4' = i\sqrt{2\pi} \sum_n{}' i^n[a_n' \cosh n\psi + a_n\psi \sinh n\psi]$$

$$= i\pi \lim_{\rho\to 0} \frac{V_5}{\rho} = i\pi \lim_{\rho \to 0} \frac{V_6}{\rho}. \tag{207}$$

As before we find that U_7, valid when z is large, is a linear combination of two independent solutions which have been established as useful in the neighborhood of $z = 1$.

The formulas for K_3 and K_4 reduce to the following when $a = -\frac{1}{2}$.

$$K_3 = -\frac{i\left(\dfrac{c}{4}\right)^{s+\rho-1} \displaystyle\sum_{n \lesseqgtr s}{}' \frac{a_n i^{s-n}(n+\rho)}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n}{2}+\rho\right)!}}{2\sqrt{2\pi}\,(-s-\rho)!\displaystyle\sum_{n\geqq s}{}' \frac{a_n i^n}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n}{2}-\rho\right)!}} \tag{208}$$

$$K_4 = \frac{i\left(\dfrac{c}{4}\right)^{-s-\rho-1} \displaystyle\sum_{n \geqq s}{}' \frac{a_n i^{n-s}(n+\rho)}{\left(\dfrac{n-s}{2}\right)!\left(-\dfrac{s+n}{2}-\rho\right)!}}{2\sqrt{2\pi}\,(s+\rho)!\displaystyle\sum_{n\lesseqgtr s}{}' \frac{a_n i^n}{\left(\dfrac{s-n}{2}\right)!\left(\dfrac{s+n}{2}+\rho\right)!}}. \tag{209}$$

When n is even let $s = 2$ for K_3 and $s = -2$ for K_4. There follows after simplification:

$$K_3' = -K_4' = K_3 \ln\frac{c}{4} + \frac{ic}{4}\sqrt{\frac{2}{\pi}}\left\{\frac{\left(C+\dfrac{1}{4}\right)a_2 - \displaystyle\sum_{n=4}^{\infty}{}' \frac{a_n}{\left(\dfrac{n}{2}\right)^2 - 1}}{\displaystyle\sum_{n=2}^{\infty}{}' a_n n} - \frac{a_2 \displaystyle\sum_{n=2}^{\infty} n\left[a_n' + a_n\sigma\left(\frac{n}{2}\right)\right]}{\left[\displaystyle\sum_{n=2}^{\infty}{}' a_n n\right]^2}\right\}. \tag{210}$$

When n is odd let $s = 1$ for K_3 and $s = -1$ for K_4. We obtain:

$$K_3' = K_4' = K_3 \ln \frac{c}{4} + \frac{1}{\sqrt{2\pi}} \left\{ \frac{\left(\frac{1}{2} - C\right) a_1 + 2 \sum_{n=3}^{\infty}{}' \frac{a_n n}{n^2 - 1}}{\sum_{n=1}^{\infty}{}' a_n} + \frac{a_1 \sum_{n=1}^{\infty}{}' \left[a_n' + a_n \sigma\left(\frac{n-1}{2}\right)\right]}{\left[\sum_{n=1}^{\infty}{}' a_n\right]^2} \right\}. \tag{211}$$

In general,

$$K_3' = -(-1)^n K_4', \tag{212}$$

$$U_7 = \frac{2}{\pi} K_3' V_3 + iK_3 \frac{V_6}{\rho}. \tag{213}$$

2.13 Recursion Formulas for a_n' when $a = -\frac{1}{2}$, $\rho \to 0$. Let $\epsilon = b - c^2/2$. The recursion formula (100) for the Mathieu equation is then

$$a_{n+2} + a_{n-2} + \frac{4}{c^2} (\epsilon - n^2) a_n = 0. \tag{214}$$

Or, upon placing

$$r_n = a_n / a_{n-2}, \tag{215}$$

it may be expanded as in Sec. 2.6 in a continued fraction

$$r_n = \frac{-c^2/4}{\epsilon - n^2 + \frac{c^2}{4} r_{n+2}} = \cfrac{-c^2/4}{\epsilon - n^2 - \cfrac{c^4/16}{\epsilon - (n+2)^2 - \cfrac{c^4/16}{\epsilon - (n+4)^2 - \cdots}}}. \tag{216}$$

Replace n by $n + \rho$, differentiate (216) with respect to ρ, and let $\rho \to 0$.

$$r_n' = \lim_{\rho \to 0} \frac{dr_{n+\rho}}{d\rho} = -\frac{4}{c^2} r_n^2 \left(2n - \frac{c^2}{4} r_{n+2}'\right) = -\frac{4}{c^2} r_n^2 \{2n + r_{n+2}^2 [2(n+2) + r_{n+4}^2 (2(n+4) + \cdots)]\}. \tag{217}$$

From (215) we obtain

$$r_n' = r_n\left(\frac{a_n'}{a_n} - \frac{a_{n-2}'}{a_{n-2}}\right). \tag{218}$$

The values of r_n can be determined from (216), whence in virtue of (217) one may consider r_n' to be a known quantity. To obtain each individual a_n', one must know one particular a_n' for a specified n, or establish an additional relationship between the a_n'. There are four cases: Even functions, l even or odd; Odd functions, l even or odd.

Even functions, l and n even. According to (101) $a_n = (-1)^n a_{-n}$, and hence $a_n = a_{-n}$. Consider the coefficients to be plotted as functions of the variable $n + \rho$. In view of the symmetry about the axis $n + \rho = 0$, it is apparent that $a_0' = 0$, $a_n' = -a_{-n}'$. Since in this case a_0 differs from zero we learn from (218) that $a_2' = a_0 r_2'$. The remaining coefficients a_n' can then be found from the general formula

$$a_n' = a_n\left(\frac{a_{n-2}'}{a_{n-2}} + \frac{r_n'}{r_n}\right). \tag{219}$$

Even functions, l and n odd. Then $a_n = -a_{-n}$. The coefficient a_n is an odd function of n and its derivative is therefore even. $a_n' = a_{-n}'$. In particular $a_1' = a_{-1}'$, $a_1 = -a_{-1}$, $r_1 = -1$. From (218) we obtain $a_1' = -a_1 r_1'/2$ and the remaining a_n' are found from (219).

Odd functions, l and n odd. According to (104) $a_n = -(-1)^n a_{-n}$, and hence $a_n = a_{-n}$, $a_n' = -a_{-n}'$. From (218) one finds $a_1' = a_1 r_1'/2$ and the remaining coefficients again follow from (219).

Odd functions, l and n even. Then $a_n = -a_{-n}$, $a_n' = a_{-n}'$, $a_0 = 0$. Upon differentiating (214) and placing $n = 2$ one obtains

$$a_2' = -\frac{2\epsilon}{c^2}\, a_0'. \tag{220}$$

From (217) we find

$$\frac{r_2'}{r_2} = -\frac{4}{c^2}\, r_2\{4 + r_4^2[8 + \cdots]\} = -\frac{a_2}{a_0}\, K, \tag{221}$$

and likewise from (218) and (220)

$$\frac{r_2'}{r_2} = -\left(\frac{2\epsilon}{c^2 a_2} + \frac{1}{a_0}\right) a_0' = -\frac{a_2}{a_0}\, K. \tag{222}$$

Since a_2 differs from zero one may multiply (222) by a_0 to obtain

$$a_0' = a_2 K, \tag{223}$$

a finite quantity. The remaining coefficients can now be determined from (219).

In view of the symmetric or anti-symmetric relations between the coefficients the second solutions of the Mathieu equation reduce to the following:

Even functions, l and n even,

$$\frac{V_8}{\rho} = \sqrt{\frac{2}{\pi}}\left[a_0\psi + 2\sum_{n=2}^{\infty}{}' i^n(a_n' \sinh n\psi + a_n\psi \cosh n\psi)\right], \tag{224}$$

and for l and n odd,

$$\frac{V_8}{\rho} = 2\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}{}' i^n(a_n' \sinh n\psi + a_n\psi \cosh n\psi). \tag{225}$$

Odd functions, l and n even,

$$\frac{V_6}{\rho} = \sqrt{\frac{2}{\pi}}\left[a_0' + 2\sum_{n=2}^{\infty}{}' i^n(a_n' \cosh n\psi + a_n\psi \sinh n\psi)\right], \tag{226}$$

and for l and n odd,

$$\frac{V_6}{\rho} = 2\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}{}' i^n(a_n' \cosh n\psi + a_n\psi \sinh n\psi). \tag{227}$$

The summations are extended over positive values of n. These solutions are identical with those found by Ince[10]. However, Ince's treatment expresses the coefficients a_n and a_n' as analytic functions of c and its application is limited to small values of c.

III. Prolate Spheroidal Functions

3.1 **Functions of the First Kind.** In the preceding sections, sixteen solutions have been found for the basic equation (7) and relations established that permit the analytic continuation of any one solution over the entire z-plane. We shall now specialize these functions for certain physical problems. Consider first wave functions of the prolate spheroid. In this case $a = m$, a positive integer, and the upper sign is chosen in Eqs. (3), (5), and (6). The V-functions are then functions of angle, or position on the surface of a spheroid, so that the variable is confined to the range $-1 \leq z \leq 1$. Radial variation is given by the U-functions, with the variable limited to the range $z \geq 1$. Those solutions that are finite at the poles $z = \pm 1$ are termed functions of the first kind. Functions of the second kind are infinite at these points.

For the standard V-function of the first kind we choose V_8, multiplied by a factor i^{-l}. We define:

$$V_{ml}^{(1)}(c, z) = i^{-l} V_8 = \sum_{n=0,1}^{\infty}{}' d_n^l T_n^m(z), \tag{228}$$

where by (92)

$$d_n^l = i^{n-l} \frac{n!}{(n+2m)!} a_n^l. \tag{229}$$

The summation starts from $n = 0$ when l is even, from $n = 1$ when l is odd, and extends over alternate positive values of n.

The coefficients will be normalized such that each spheroidal function reduces to the corresponding spherical function as $c \to 0$. It proves more convenient to carry out the normalization at $z = 0$ than at the poles $z = \pm 1$. Thus it is desired that when l is even,

$$V_{ml}^{(1)}(c, 0) = \sum_{n=0,1}^{\infty}{}' d_n^l T_n^m(0) = T_l^m(0), \tag{230}$$

whence

$$V_{ml}^{(1)}(0, z) = T_l^m(z), \tag{231}$$

and when l is odd,

$$\frac{d}{dz} V_{ml}^{(1)}(c, z)\bigg|_{z=0} = \frac{d}{dz} T_l^m(z)\bigg|_{z=0}. \tag{232}$$

For even values of l one has

$$T_l^m(0) = \frac{i^l 2^m \left(\frac{l-1}{2} + m\right)!}{\sqrt{\pi}\left(\frac{l}{2}\right)!}, \tag{233}$$

while for odd values

$$\frac{d}{dz} T_l^m(z)\bigg|_{z=0} = \frac{i^{l-1} 2^{m+1} \left(\frac{l}{2} + m\right)!}{\sqrt{\pi}\left(\frac{l-1}{2}\right)!}. \tag{234}$$

Hence

$$\sum_{n=0}^{\infty}{}' \frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n}{2}+m\right)!} a_n = \frac{2^{2m}\left(\frac{l-1}{2}+m\right)!}{\left(\frac{l}{2}\right)!}, \tag{235}$$

when l is even, and

$$\sum_{n=1}^{\infty}{}' \frac{\left(\frac{n}{2}\right)!}{\left(\frac{n-1}{2}+m\right)!} a_n = \frac{2^{2m}\left(\frac{l}{2}+m\right)!}{\left(\frac{l-1}{2}\right)!} \tag{236}$$

when l is odd. This fixes the absolute magnitudes of the coefficients a_n

The corresponding solution of the *wave equation* (1) will be denoted $S_{ml}^{(1)}(c, \eta)$.

$$\begin{aligned} S_{ml}^{(1)}(c, \eta) &= (1-\eta^2)^{m/2} V_{ml}^{(1)}(c, \eta) \\ &= \sum_{n=0,1}^{\infty}{}' d_n^l \, P_{n+m}^m(\eta). \end{aligned} \tag{237}$$

For the standard U-function of the first kind we choose U_1 with an appropriate normalization factor.

$$\begin{aligned} U_{ml}^{(1)}(c, z) &= \kappa_l \sqrt{\frac{\pi}{2}}\, U_1 \\ &= \kappa_l (cz)^{-m} \sum_{n=0,1}^{\infty}{}' a_n^l \, j_{n+m}(cz), \end{aligned} \tag{238}$$

where κ_l is the normalization factor, and

$$j_{n+m}(cz) = \sqrt{\frac{\pi}{2cz}}\, J_{n+m+\frac{1}{2}}(cz). \tag{239}$$

The corresponding solution of the wave equation is then defined as

$$R_{ml}^{(1)}(c, \xi) = (\xi^2 - 1)^{m/2} U_{ml}^{(1)}(c, \xi). \tag{240}$$

When $c\xi \to \infty$,

$$j_{n+m}(c\xi) \to \frac{i^{n-l}}{c\xi} \sin\left(c\xi - \frac{l+m}{2}\pi\right), \tag{241}$$

so that

$$R_{ml}^{(1)}(c, \xi) \to \frac{\kappa_l}{c^{m+1}\xi} \sin\left(c\xi - \frac{l+m}{2}\pi\right) \sum_{n=0,1}^{\infty}{}' a_n^l \, i^{n-l}. \tag{242}$$

(Strictly (242) does not follow directly from (241), since (241) is valid only when the argument is very much larger than the order. A more careful examination of the asymptotic behavior of $a_n^l j_{n+m}(c\xi)$ on the grounds of Sec. 2.7 leads to the same result.) The normalization factor will now be chosen such that for very large values of $c\xi$,

$$R_{ml}^{(1)}(c, \xi) \to \frac{1}{c\xi} \sin\left(c\xi - \frac{l+m}{2}\pi\right). \tag{243}$$

Thus κ_l is determined by

$$\kappa_l = c^m \Big/ \sum_{n=0,1}^{\infty}{}' a_n^l i^{n-l}. \tag{244}$$

The functions $V_{ml}^{(1)}(c, z)$ and $U_{ml}^{(1)}(c, z)$ are different representations of one and the same solution. From the results of Sec. 2.9 the following relations may be established:

For l even,

$$U_{ml}^{(1)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, a_0}{2^{m+1}(m + \frac{1}{2})!} \frac{V_{ml}^{(1)}(c, z)}{V_{ml}^{(1)}(c, 0)}, \tag{245}$$

and for l odd,

$$U_{ml}^{(1)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, c a_1}{2^{m+2}(m + \frac{3}{2})!} \frac{V_{ml}^{(1)}(c, z)}{\left[\dfrac{d}{dz} V_{ml}^{(1)}(z)\right]_{z=0}}. \tag{246}$$

From the orthogonal properties of the Gegenbauer functions it may be shown that

$$\int_{-1}^{1} V_{ml}^{(1)}(c, z) V_{ml'}^{(1)}(c, z)(1 - z^2)^m\, dz = \int_{-1}^{1} S_{ml}^{(1)}(c, \eta) S_{ml'}^{(1)}(c, \eta)\, d\eta$$

$$= \begin{cases} 0, & l \neq l' \\ 2 \displaystyle\sum_{n=0,1}^{\infty}{}' (a_n^l)^2 \frac{n!}{(n + 2m)!(2n + 2m + 1)}, & l = l'. \end{cases} \tag{247}$$

3.2 **Functions of the Second Kind.** These are constructed from functions of the second group tabulated in Sec. 2.10. They are characterized by logarithmic singularities at the points $z = \pm 1$. We define:

$$\begin{aligned} U_{ml}^{(2)}(c, z) &= \kappa_l \sqrt{\frac{\pi}{2}}\, U_5 \\ &= \kappa_l (cz)^{-m} \sum_{n=0,1}^{\infty}{}' a_n^l\, n_{n+m}(cz), \end{aligned} \tag{248}$$

where $n_{n+m}(cz)$ is a spherical Bessel function of the second kind,

$$n_{n+m}(cz) = \sqrt{\frac{\pi}{2cz}}\, N_{n+m+\frac{1}{2}}(cz). \tag{249}$$

The corresponding solution of the wave equation is then

$$R_{ml}^{(2)}(c, \xi) = (\xi^2 - 1)^{m/2} U_{ml}^{(2)}(c, \xi). \tag{250}$$

As $c\xi \to \infty$,

$$R_{ml}^{(2)}(c, \xi) \to -\frac{1}{c\xi} \cos\left(c\xi - \frac{l+m}{2}\pi\right). \tag{251}$$

It is desirable to define $V_{ml}^{(2)}(c, z)$ in such a way that

$$\frac{U_{ml}^{(1)}(c, z)}{V_{ml}^{(1)}(c, z)} = \frac{U_{ml}^{(2)}(c, z)}{V_{ml}^{(2)}(c, z)} \tag{252}$$

and to obtain for it a representation valid in the neighborhood of $z = 1$. From Sec. 2.10 one finds

$$U_{ml}^{(2)}(c, z) = \kappa_l \sqrt{\frac{\pi}{2}}\, (-1)^{l+m+1} K_2 V_4. \tag{253}$$

This is combined with (252), and leads, when l is even, to

$$V_{ml}^{(2)}(c, z) = -\frac{i^{2m+l}(4m^2 - 1)}{ca_0(\rho - 2m)!\, a_{-2m}} [V_{ml}^{(1)}(c, 0)]^2 V_4, \tag{254}$$

or

$$U_{ml}^{(2)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, a_0}{2^{m+1}(m + \frac{1}{2})!} \frac{V_{ml}^{(2)}(c, z)}{V_{ml}^{(1)}(c, 0)}. \tag{255}$$

Likewise, when l is odd,

$$V_{ml}^{(2)}(c, z) = -i^{2m-l} \frac{(4m^2 - 1)(4m^2 - 9)}{c^3 a_1(\rho + 1 - 2m)!\, a_{1-2m}} \left[\frac{d}{dz} V_{ml}^{(1)}(c, z)\right]_{z=0}^2 V_4, \tag{256}$$

and

$$U_{ml}^{(2)}(c, z) = \frac{\kappa_l \sqrt{\pi}\, ca_1}{2^{m+2}(m + \frac{3}{2})!} \frac{V_{ml}^{(2)}(c, z)}{\left[\dfrac{d}{dz} V_{ml}^{(1)}(c, z)\right]_{z=0}}. \tag{257}$$

The Wronskian of the two independent solutions for large values of cz reduces to

$$U_{ml}^{(1)} \frac{d}{dz} U_{ml}^{(2)} - U_{ml}^{(2)} \frac{d}{dz} U_{ml}^{(1)} = c^{-1} z^{-2m-2} \tag{258}$$

By analogy with the Bessel functions one may construct solutions corresponding to the Hankel functions.

$$U_{ml}^{(3)}(c, z) = U_{ml}^{(1)}(c, z) + iU_{ml}^{(2)}(c, z) \tag{259}$$

$$U_{ml}^{(4)}(c, z) = U_{ml}^{(1)}(c, z) - iU_{ml}^{(2)}(c, z) \tag{260}$$

$$V_{ml}^{(3)}(c, z) = V_{ml}^{(1)}(c, z) + iV_{ml}^{(2)}(c, z) \tag{261}$$

$$V_{ml}^{(4)}(c, z) = V_{ml}^{(1)}(c, z) - iV_{ml}^{(2)}(c, z). \tag{262}$$

Obviously the ratio of any $U_{ml}(c, z)$ to the $V_{ml}(c, z)$ function of the same kind and order is constant and equal to

$$\begin{aligned} &\frac{\kappa_l\sqrt{\pi}\,a_0}{2^{m+1}(m+\frac{1}{2})!\,V_{ml}^{(1)}(c, 0)} && \text{for } l \text{ even,} \\ &\frac{\kappa_l\sqrt{\pi}\,ca_1}{2^{m+2}(m+\frac{3}{2})!\left[\dfrac{d}{dz}V_{ml}^{(1)}(c, z)\right]_{z=0}} && \text{for } l \text{ odd.} \end{aligned} \tag{263}$$

The corresponding wave functions are $R_{ml}^{(3)}(c, \xi)$ and $R_{ml}^{(4)}(c, \xi)$, whose asymptotic behavior when $c\xi$ is very large is expressed by

$$R_{ml}^{(3)}(c, \xi) \rightarrow \frac{1}{c\xi}e^{i\left(c\xi - \frac{l+m+1}{2}\pi\right)}, \tag{264}$$

$$R_{ml}^{(4)}(c, \xi) \rightarrow \frac{1}{c\xi}e^{-i\left(c\xi - \frac{l+m+1}{2}\pi\right)}. \tag{265}$$

If by $R_{ml}(c, \xi)$ and $S_{ml}(c, \eta)$ written without superscript one understands any solution of the four kinds, then the prolate spheroidal wave functions of Eq. (3) are

$$W = R_{ml}(c, \xi)\, S_{ml}(c, \eta)\, e^{\pm im\varphi}. \tag{266}$$

IV. Oblate Spheroidal Functions.

4.1 **Definition.** Let $V_{ml}(c, z)$ and $U_{ml}(c, z)$ be standard V- and U-functions as defined in III. The kind of function will normally be indicated by a superscript. Then $V_{ml}(-ic, z)$ satisfies

$$(1 - z^2)V'' - 2(m + 1)zV' + (b_l + c^2z^2)V = 0, \tag{267}$$

while $U_{ml}(-ic, iz)$ is a solution of

$$(z^2 + 1)U'' + 2(m + 1)zU' - (b_l - c^2z^2)U = 0. \tag{268}$$

These in turn are the equations of oblate spheroidal wave functions Eqs. (3)–(6).

$$S_{ml}(-ic, \eta) = (1 - \eta^2)^{m/2} V_{ml}(-ic, \eta), \tag{269}$$

$$R_{ml}(-ic, i\xi) = i^m(\xi^2 + 1)^{m/2}\, U_{ml}(-ic, i\xi), \tag{270}$$

$$W = R_{ml}(-ic, i\xi)\, S_{ml}(-ic, \eta)\, e^{\pm im\varphi}. \tag{271}$$

In the oblate as well as the prolate case the normalization of the coefficients a_n^l is such that in the equatorial plane transverse to the axis of revolution, $\eta = \cos\theta = 0$, the spheroïdal S_{ml}-function reduces to the value of the corresponding spherical function.

V. Mathieu Functions

5.1 **Mathieu Equations.** In elliptic coordinates both the angular and radial parts of the wave function (9) satisfy an equation of the form

$$(z^2 - 1)U'' + zU' + (c^2z^2 - b)U = 0. \tag{272}$$

In the case of the angle functions z is confined to the range $|z| \leqq 1$, while for the radial functions $z \geqq 1$. Upon making a change of independent variable,

$$z = \cos\varphi = \cosh\psi, \qquad \psi = i\varphi, \tag{273}$$

one obtains

$$\frac{d^2V}{d\varphi^2} + (b - c^2\cos^2\varphi)V = 0, \tag{274}$$

$$\frac{d^2U}{d\psi^2} - (b - c^2\cosh^2\psi)U = 0. \tag{275}$$

Independent solutions of these equations will be defined below in a form which the authors believe to be the simplest and the most useful for application to physical problems. Apart from a slight change in the notation for the radial functions, these are the definitions proposed earlier by Stratton and Morse[6,11].

5.2 **Even Functions of the First Kind.** In this case the expansion coefficients are subject to the condition $a_n = (-1)^n a_{-n}$. Let

$$\begin{aligned} D_0 &= i\sqrt{\frac{\pi}{2}}\, a_0, \\ D_n &= i^{n+1}\sqrt{2\pi}\, a_n, \qquad (n > 0). \end{aligned} \tag{276}$$

[11] P. M. Morse, Proc. Nat. Acad. Sci., *6*, 56-62, 1935.

Then for both even and odd values of l Eqs. (172) and (173) reduce to

$$V_1 = \sum_{n=0,1}^{\infty}{}' D_n \cosh n\psi, \tag{277}$$

$$U_1 = \sqrt{\frac{2}{\pi}} \sum_{n=0,1}^{\infty}{}' i^{-n-1} D_n J_n(cz). \tag{278}$$

We take V_1 to be the standard V- or angle function of the first kind and define:

$$Se_l^{(1)}(c, \cos\varphi) = \sum_{n=0,1}^{\infty}{}' D_n^l \cos n\varphi. \tag{279}$$

This function will be normalized at the point $\varphi = 0$, or $z = 1$, such that

$$Se_l^{(1)}(c, 1) = 1, \tag{280}$$

or

$$\sum_{n=0,1}^{\infty}{}' D_n^l = 1. \tag{281}$$

For the standard U- or radial function of the first kind we choose[12]

$$Je_l(c, z) = \kappa_l U_1(c, z), \tag{282}$$

where κ_l is a proportionality factor determined by the normalization. It is desired that as $cz \to \infty$,

$$Je_l(c, z) \to \frac{1}{\sqrt{cz}} \cos\left(cz - \frac{2l+1}{4}\pi\right). \tag{283}$$

Thus it follows that

$$\kappa_l = i^{l+1}\pi/2, \tag{284}$$

and

$$Je_l(c, z) = \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' i^{l-n} D_n^l J_n(cz). \tag{285}$$

Since $U_1 = K_1 V_1$, one has

$$Je_l(c, z) = \kappa_l K_1 \, Se_l^{(1)}(c, z). \tag{286}$$

[12] The notation $Re_l^{(1)}$, $Re_l^{(2)}$, $Ro_l^{(1)}$, $Ro_l^{(2)}$ employed in an earlier paper (Proc. Nat. Acad. Sci., *6*, 51, 1935) for the even and odd radial functions of the first and second kinds has been replaced by Je_l, Ne_l, Jo_l, No_l to eliminate a superscript and to emphasize a relationship with the corresponding solutions of Bessel's equation.

Upon replacing a_n by D_n in (152), we obtain

$$Se_l^{(1)}(c, z) = \sqrt{2\pi}\, \lambda_l^{(e)} Je_l(c, z), \tag{287}$$

where

$$\begin{aligned} \lambda_l^{(e)} &= \frac{1}{\pi} \sum_{n=0}^{\infty}{}' i^{n-l} \frac{D_n^l}{D_0^l}, && (l \text{ even}) \\ \lambda_l^{(e)} &= \frac{2}{c\pi} \sum_{n=1}^{\infty}{}' n i^{n-l} \frac{D_n^l}{D_1^l} && (l \text{ odd}). \end{aligned} \tag{288}$$

5.3 **Even Functions of the Second Kind.** The radial functions of the second kind are obtained by replacing $J_n(cz)$ in (285) by the Bessel function of the second kind $N_n(cz)$.

$$\begin{aligned} Ne_l(c, z) &= \kappa_l U_5 \\ &= \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' i^{l-n} D_n^l N_n(c, z). \end{aligned} \tag{289}$$

As $cz \to \infty$,

$$Ne_l(c, z) \to \frac{1}{\sqrt{cz}} \sin\left(cz - \frac{2l+1}{4}\pi\right). \tag{290}$$

The convergence of (289) is limited to the region $|z| > 1$. To obtain expressions valid in the neighborhood of $z = 1$ or $z = 0$ one must replace U_5 by its equivalent expansion (198) in terms of hyperbolic functions. We shall define

$$\begin{aligned} Se_l^{(2)}(c, z) &= U_5/K_1 \\ &= \frac{2}{\pi}\left\{\frac{K_1'}{K_1} Se_l^{(1)}(c, z) + \sum_{n=0,1}^{\infty}{}' [D_n' \sinh n\psi + \psi D_n \cosh n\psi]\right\} \end{aligned} \tag{291}$$

where

$$D_n' = i^{n+1} \sqrt{2\pi}\, a_n', \qquad (n > 0), \tag{292}$$

$$\frac{K_1'}{K_1} = ln \frac{c}{4} + C - \sum_{n=2}^{\infty}{}' \frac{i^n D_n}{n D_0} - \frac{\displaystyle\sum_{n=2}^{\infty}{}' i^n \left\{D_n' + D_n\left[\sigma\left(\frac{n}{2} - 1\right) + \frac{1}{n}\right]\right\}}{\displaystyle\sum_{n=0}^{\infty}{}' i^n D_n} \tag{293}$$

for l even, and

$$\frac{K_1'}{K_1} = ln\frac{c}{4} + C + \frac{1}{2} - 2\sum_{n=3}^{\infty}{}' \frac{i^{n-1}D_n}{(n^2-1)D_1} - \frac{\sum_{n=1}^{\infty} i^n n\left\{D_n' + D_n\left[\sigma\left(\frac{n-1}{2}\right) + \frac{1}{n}\right]\right\}}{\sum_{n=1}^{\infty} i^n nD_n} \tag{294}$$

for l odd. C and $\sigma(x)$ are defined as in (195).

It is obvious that

$$Se_l^{(2)}(c, z) = \sqrt{2\pi}\,\lambda_l^{(e)}\,Ne_l(c, z). \tag{295}$$

The Wronskian of $Je_l(c, \cosh\psi)$ and $Ne_l(c, \cosh\psi)$ is unity.

$$Je_l\frac{d}{d\psi}Ne_l - Ne_l\frac{d}{d\psi}Je_l = 1. \tag{296}$$

Through combinations of these functions one may define Mathieu functions of the third and fourth kinds.

$$He_l^{(1)}(c, z) = Je_l(c, z) + iNe_l(c, z), \tag{297}$$

$$He_l^{(2)}(c, z) = Je_l(c, z) - iNe_l(c, z). \tag{298}$$

The notation corresponds to that of the Hankel functions. Similar functions can be constructed from $Se_l^{(1)}(c, z)$ and $Se_l^{(2)}(c, z)$. When $cz \to \infty$,

$$He_l^{(1)}(c, z) \to \frac{1}{\sqrt{cz}}e^{i\left(cz - \frac{2l+1}{4}\pi\right)}, \tag{299}$$

$$He_l^{(2)}(c, z) \to \frac{1}{\sqrt{cz}}e^{-i\left(cz + \frac{2l+1}{4}\pi\right)}. \tag{300}$$

5.4 **Odd Functions of the First Kind.** These are constructed from U_3- and V_3-functions subject to the condition $a_n = -(-1)^n a_{-n}$. Let

$$F_n = -i^n\sqrt{2\pi}\,a_n. \tag{301}$$

Then according to (179) and (180)

$$V_3 = -i\sum_{n=1,2}^{\infty}{}' F_n \sinh n\psi, \tag{302}$$

$$U_3 = -\frac{\sqrt{z^2-1}}{cz}\sqrt{\frac{2}{\pi}}\sum_{n=1,2}^{\infty}{}' ni^{-n}F_nJ_n(cz). \tag{303}$$

We take V_3 to be the standard V- or angle function of the first kind and define:

$$So_l^{(1)}(c, \cos\varphi) = \sum_{n=1,2}^{\infty}{}' F_n^l \sin n\varphi, \tag{304}$$

normalized at $z = 1$, or $\varphi = 0$, such that

$$\left[\frac{d}{d\varphi} So_l^{(1)}(c, \cos\varphi)\right]_{\varphi=0} = 1. \tag{305}$$

From this it follows that the F_n^l must be such that

$$\sum_{n=1,2}^{\infty}{}' nF_n^l = 1. \tag{306}$$

For the standard U- or radial function of the first kind we choose

$$Jo_l(c, z) = \kappa_l U_3(c, z). \tag{307}$$

The factor κ_l is fixed by the condition that when $cz \to \infty$,

$$Jo_l(c, z) \to \frac{1}{\sqrt{cz}}\cos\left(cz - \frac{2l+1}{4}\pi\right), \tag{308}$$

whence

$$\kappa_l = -i^l\pi c/2, \tag{309}$$

$$Jo_l(c, z) = \frac{\sqrt{z^2-1}}{z}\sqrt{\frac{2}{\pi}}\sum_{n=1,2}^{\infty}{}' ni^{l-n}F_n^lJ_n(cz). \tag{310}$$

Since $U_3 = K_3V_3$, one has

$$Jo_l(c, z) = \kappa_l K_3\, So_l^{(1)}(c, z). \tag{311}$$

Upon replacing a_n by F_n in (165) and (169), we obtain

$$iSo_l^{(1)}(c, z) = \sqrt{2\pi}\,\lambda_l^{(o)}\, Jo_l(c, z), \tag{312}$$

where

$$\begin{aligned} \lambda_l^{(o)} &= \frac{4}{\pi c^2}\sum_{n=2}^{\infty}{}' ni^{n-l}\frac{F_n}{F_2}, && (l \text{ even}) \\ \lambda_l^{(o)} &= \frac{2}{\pi c}\sum_{n=1}^{\infty}{}' i^{n-l}\frac{F_n}{F_1}, && (l \text{ odd}). \end{aligned} \tag{313}$$

5.5 **Odd Functions of the Second Kind.** The radial function of the second kind is defined by

$$
\begin{aligned}
No_l(c, z) &= \kappa_l U_7 \\
&= \frac{\sqrt{z^2 - 1}}{z} \sqrt{\frac{\pi}{2}} \sum_{n=1,2}^{\infty}{}' \, n i^{l-n} F_n N_n(cz),
\end{aligned} \tag{314}
$$

wherein U_7 is given by (181). As $cz \to \infty$,

$$
No_l(c, z) \to \frac{1}{\sqrt{cz}} \sin\left(cz - \frac{2l+1}{4}\pi\right). \tag{315}
$$

The convergence of (314) is rapid only when cz is large. In the neighborhood of $z = \pm 1$ one must replace this expansion by its equivalent series (213) in terms of hyperbolic functions. Again by definition,

$$
\begin{aligned}
So_l^{(2)}(c, z) &= U_7/K_3 \\
&= \frac{2}{\pi}\left\{\frac{K_3'}{K_3} So_l^{(1)}(c, z) - i \sum_{n=0,1}^{\infty}{}' [F_n' \cosh n\psi + F_n \psi \sinh n\psi]\right\},
\end{aligned} \tag{316}
$$

where

$$
F_0' = -\sqrt{\frac{\pi}{2}}\, a_0', \qquad F_n' = -\sqrt{2\pi}\, i^n a_n', \tag{317}
$$

$$
\frac{K_3'}{K_3} = ln\frac{c}{4} + C + \frac{1}{4} + 4\sum_{n=4}^{\infty}{}' \frac{i^n F_n}{(n^2 - 4)F_2} - \frac{\sum_{n=2}^{\infty}{}' n i^n \left[F_n' + F_n \sigma\left(\frac{n}{2}\right)\right]}{\sum_{n=2}^{\infty} n i^n F_n} \tag{318}
$$

for l even, and

$$
\frac{K_3'}{K_3} = ln\frac{c}{4} + C - \frac{1}{2} + 2\sum_{n=3}^{\infty}{}' \frac{n i^{n+1} F_n}{(n^2 - 1)F_1} - \frac{\sum_{n=1}^{\infty}{}' i^{n+1}\left[F_n' + F_n \sigma\left(\frac{n-1}{2}\right)\right]}{\sum_{n=1}^{\infty}{}' i^{n+1} F_n} \tag{319}
$$

when l is odd. It is again apparent that

$$
i So_l^{(2)}(c, z) = \sqrt{2\pi}\, \lambda_l^{(o)} No_l(c, z). \tag{320}
$$

The Wronskian of the functions $Jo_l(c, \cosh \psi)$ and $No_l(c, \cosh \psi)$ is unity.

$$Jo_l \frac{d}{d\psi} No_l - No_l \frac{d}{d\psi} Jo_l = 1. \tag{321}$$

Odd Mathieu functions of the third and fourth kinds corresponding, to the Hankel functions are constructed in the usual manner.

$$Ho_l^{(1)}(c, z) = Jo_l(c, z) + iNo_l(c, z), \tag{322}$$

$$Ho_l^{(2)}(c, z) = Jo_l(c, z) - iNo_l(c, z). \tag{323}$$

When $cz \to \infty$,

$$Ho_l^{(1)}(c, z) \to \frac{1}{\sqrt{cz}} e^{i\left(cz - \frac{2l+1}{4}\pi\right)}, \tag{324}$$

$$Ho_l^{(2)}(c, z) \to \frac{1}{\sqrt{cz}} e^{-i\left(cz - \frac{2l+1}{4}\pi\right)}. \tag{325}$$

Professor P. M. Morse and the authors are indebted to the Kennelly Fund of the Massachusetts Institute of Technology for funds to subsidize the computation of numerical values for the functions defined in this paper. In the case of the elliptic cylinder functions the coefficients D_n^l and F_n^l, and the separation constants b_l have been calculated for values of c over the range 0 to 4.5 at intervals of 0.2, and for $l = 1, 2, 3, 4$. The separation constants for both prolate and oblate spheroidal functions have been computed for values of c over the range 0 to 5.0 at intervals of 0.2, and for m and $l = 0, 1, 2, 3$; $m + l \leqq 3$. The coefficients d_m^l likewise have been calculated for the same values. These tables eventually will be published. Until then they may be obtained in mimeographed form by application to the Department of Physics at M. I. T.

The authors wish to acknowledge their indebtedness to Dr. R. Albagli Hutner who has verified all the formulas employed in this paper and who has supervised the computation of expansion coefficients and separation constants. Dr. Hutner contributed also the method described in Sec. 2.13 for obtaining the coefficients a_n' in the case of Mathieu functions of the second kind.

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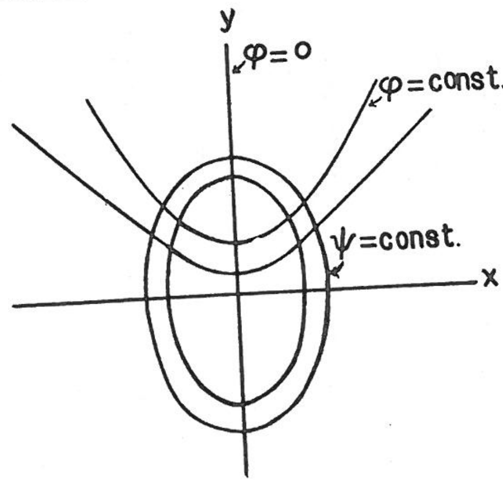
This introduction to the tables merely lists the formulas for which the tables of coefficients are useful. They are based on the theory contained in the preceding article by L. J. Chu and J. A. Stratton. For the sake of clarity one fact should be emphasized; namely, that Chu and Stratton, when discussing the coefficients of the spheroidal functions, deal with those, a_n^{ℓ}, which occur in the $R_{m\ell}^{(2)}$ solution, while the coefficients listed in the tables are the ones, d_n^{ℓ}, which occur in the $S_{m\ell}^{(1)}$ solution.

$$a_n^{\ell} = i^{\ell-n} d_n^{\ell} \frac{(n+2m)!}{n!}$$

ELLIPTIC CYLINDER FUNCTIONS



Coordinates:

$$x = \frac{d}{2} \cos \varphi \cosh \psi$$

$$y = \frac{d}{2} \sin \varphi \sinh \psi$$

$$0 \leq \varphi \leq \pi$$

$$0 \leq \psi < \infty$$

d = distance between foci

Wave equation yields

$$\frac{d^2S}{d\varphi^2} + (b_\ell - c^2 \cos^2 \varphi)S = 0$$

$$\frac{d^2R}{d\psi^2} + (c^2\cosh^2\psi - b_\ell)R = 0$$

If $z = \cos\varphi = \cosh\psi$, and F = S or R, each of the above equations transforms into

$$(1-z^2)F'' - zF' + (b_\ell - c^2z^2)F = 0$$

which is the form given in the paper of Chu and Stratton.

<u>The Even Angular and Radial Functions</u> for a given value of $b_\ell(c)$.

1. Angular Function

$$Se_\ell(c,\cos\varphi) = \sum_{n=0,1}^{\infty}{}' D_n^\ell \cos(n\varphi)$$

where

$$\sum_{n=0,1}^{\infty}{}' D_n^\ell = 1$$

$$Se_\ell(c,1) = 1$$

Primed summations are over even values of n if ℓ is even, over odd values of n if ℓ is odd.

$$\int_0^{2\pi} Se_\ell(c,\cos\varphi)\, Se_k(c,\cos\varphi)\,d\varphi = \begin{cases} 0, & k \neq \ell \\ N_\ell, & k = \ell \end{cases}$$

$$N_\ell = \pi\left[2(D_0^\ell)^2 + \sum_{n=2}^{\infty}{}' (D_n^\ell)^2\right] \qquad \ell = \text{even}$$

$$N_\ell = \pi\sum_{n=1}^{\infty}{}' (D_n^\ell)^2 \qquad \ell = \text{odd}$$

2. Radial Functions.

$$Je_\ell(c,z) = \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' i^{\ell-n} D_n^\ell J_n(cz) \xrightarrow[cz\to\infty]{} \frac{1}{\sqrt{cz}} \cos\left(cz - \frac{2m+1}{4}\pi\right)$$

$$Ne_\ell(c,z) = \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' i^{\ell-n} D_n^\ell N_n(cz) \xrightarrow[cz\to\infty]{} \frac{1}{\sqrt{cz}} \sin\left(cz - \frac{2m+1}{4}\pi\right)$$

See Jahnke and Emde for a definition of $N_n(x)$.

For $z=\cosh\psi$ equal or almost equal to one, the solution Ne_ℓ may be evaluated by means of the following formulas:

$\underline{\ell = \text{even}}$:

$$Ne_\ell(c,\cosh\psi)=i^\ell\sqrt{\frac{2}{\pi}}\left\{\frac{D_0^\ell}{E_\ell}\sum_{n=0}^{\infty}{}' D_n^{\ell\,\prime}\sinh n\psi + \left(L_\ell+\frac{D_0^\ell\psi}{E_\ell}\right)\sum_{n=0}^{\infty}{}' D_n^\ell\cosh n\psi\right\}$$

where

$$E_\ell=\sum_{n=0}^{\infty}{}' i^n D_n^\ell$$

$$L_\ell=\frac{D_0^\ell\left[\ln\left(\frac{c}{4}\right)+C\right]-\sum\limits_{n=2}^{\infty}{}'\frac{D_n^\ell i^n}{n}}{E_\ell}-\frac{D_0^\ell\sum\limits_{n=2}^{\infty}{}' i^n\left\{D_n^{\ell\,\prime}+D_n^\ell\left[\frac{1}{n}+\sigma\left(\frac{n}{2}-1\right)\right]\right\}}{E_\ell^2}$$

See Note (A) at end of section on Elliptic Cylinder Functions for evaluation of C, $\sigma(n)$ and $D_n^{\ell\,\prime}$.

$\underline{\ell = \text{odd}}$:

$$Ne_\ell(c,\cosh\psi)=\frac{ci^{\ell+1}}{\sqrt{2\pi}}\left\{\frac{D_1^\ell}{E_\ell}\sum_{n=1}^{\infty}{}' D_n^{\ell\,\prime}\sinh n\psi+\left(L_\ell+\frac{D_1^\ell\psi}{E_\ell}\right)\sum_{n=1}^{\infty}{}' D_n^\ell\cosh n\psi\right\}$$

where

$$E_\ell=\sum_{n=1}^{\infty}{}' n i^{n+1} D_n^\ell$$

$$L_\ell=\frac{D_1\left[C+\frac{1}{2}+\ln\left(\frac{c}{4}\right)\right]-2\sum\limits_{n=3}^{\infty}{}'\frac{D_n i^{n-1}}{(n^2-1)}}{E_\ell}-\frac{D_1\sum\limits_{n=1}^{\infty}{}' i^{n+1}\left\{nD_n'+D_n\left[1+n\sigma\left(\frac{n-1}{2}\right)\right]\right\}}{E_\ell^2}$$

See Note (A) at end of section on Elliptic Cylinder Functions for evaluation of C, $\sigma(n)$ and $D_n^{\ell\,\prime}$.

3. Joining Factors.

$$Se_\ell(c,z) = \sqrt{2\pi}\,\lambda_\ell\, Je_\ell(c,z)$$

$$\lambda_\ell = \frac{1}{\pi} \sum_{n=0}^{\infty}{}' \, i^{n-\ell} \frac{D_n^\ell}{D_1^\ell} \qquad \ell = \text{even}$$

$$\lambda_\ell = \frac{2}{c\pi} \sum_{n=1}^{\infty}{}' \, n i^{n-\ell} \frac{D_n^\ell}{D_1^\ell} \qquad \ell = \text{odd}$$

4. Wronskian and Special Values of the Radial Functions.

$$Je_\ell(c,\cosh\psi)\frac{d}{d\psi}Ne_\ell(c,\cosh\psi) - Ne_\ell(c,\cosh\psi)\frac{d}{d\psi}Je_\ell(c,\cosh\psi) = 1$$

At $z=\cosh\psi \longrightarrow 1$:

$$Je_\ell(c,\cosh\psi) \longrightarrow \frac{1}{\sqrt{2\pi}\,\lambda_\ell}$$

$$\frac{d}{d\psi} Je_\ell(c,\cosh\psi) \longrightarrow 0$$

$$Ne_\ell(c,\cosh\psi) \longrightarrow -\sqrt{2\pi}\,\mu_\ell$$

$$\frac{d}{d\psi} Ne_\ell(c,\cosh\psi) \longrightarrow \sqrt{2\pi}\,\lambda_\ell$$

The quantity μ_ℓ is evaluated by means of the second forms given above for the Ne_ℓ solutions.

<u>The Odd Angular and Radial Functions</u> for a given value of $b'_\ell(c)$.

1. Angular Function.

$$So_\ell(c,\cos\varphi)=\sum_{n=1,2}^{\infty}{}' F_n^\ell \sin(n\varphi)$$

where

$$\sum_{n=1,2}^{\infty}{}' nF_n^\ell =1$$

$$\frac{d}{d\varphi} So_\ell(c,\cos\varphi) \xrightarrow[\varphi\to 0]{} 1$$

$$\int_0^{2\pi} So_\ell(c,\cos\varphi)\, So_k(c,\cos\varphi)d\varphi= \begin{cases} 0 , & k\neq \ell \\ N'_\ell , & k=\ell \end{cases}$$

$$N'_\ell = \pi \sum_{n=1,2}^{\infty}{}' (F_n^\ell)^2$$

2. Radial Functions.

$$Jo_\ell(c,z)=\sqrt{\frac{\pi}{2}}\ \frac{\sqrt{z^2-1}}{z} \sum_{n=1,2}^{\infty}{}' i^{n-\ell} nF_n^\ell J_n(cz) \xrightarrow[cz\to\infty]{} \frac{1}{\sqrt{cz}}\cos(cz-\frac{2m+1}{4}\pi)$$

$$No_\ell(c,z)=\sqrt{\frac{\pi}{2}}\ \frac{\sqrt{z^2-1}}{z} \sum_{n=1,2}^{\infty}{}' i^{n-\ell} nF_n^\ell N_n(cz) \xrightarrow[cz\to\infty]{} \frac{1}{\sqrt{cz}}\sin(cz-\frac{2m+1}{4}\pi)$$

Solutions of $No_\ell(c,z)$ for $z = \cos\psi$ equal or almost equal to one are:

ℓ = even :

$$No_\ell(c,\cosh\psi)=\frac{c^2 i^\ell}{2\sqrt{2\pi}}\left\{\frac{F_2^\ell}{E'_\ell}\sum_{n=0}^{\infty}{}' F_n^{\ell\prime}\cosh n\psi+\left(L'_\ell+\frac{F_2^\ell\psi}{E'_\ell}\right)\sum_{n=2}^{\infty}{}' F_n^\ell \sinh n\psi\right\}$$

where

$$E'_\ell=\sum_{n=2}^{\infty}{}' nF_n^\ell i^n$$

$$L'_\ell=\frac{F_2^\ell\left[\ln\left(\frac{c}{4}\right)+C+\frac{1}{4}\right]+4\sum_{n=4}^{\infty}{}'\frac{F_n^\ell i^n}{(n^2-4)}}{E'_\ell}-\frac{F_2\sum_{n=2}^{\infty}{}' n i^n\left[F_n^{\ell\prime}+F_n^\ell\,\sigma\left(\frac{n}{2}\right)\right]}{(E')^2}$$

See Note (A) at end of section on Elliptic Cylinder Functions for evaluation of C, $\sigma(n)$, and $F_n^{\ell\prime}$.

ℓ= odd :

$$No_\ell(c,\ \cosh\psi)=\frac{c i^{\ell+1}}{\sqrt{2\pi}}\left\{\frac{F_1^\ell}{E'_\ell}\sum_{n=1}^{\infty}{}' F_n^{\ell\prime}\cosh n\psi+\left(L'+\frac{F_1^\ell\psi}{E'_\ell}\right)\sum_{n=1}^{\infty}{}' F_n^\ell \sinh n\right.$$

where

$$E'_\ell=\sum_{n=1}^{\infty}{}' i^{n+1}F_n^\ell$$

$$L'_\ell=\frac{F_1^\ell\left[\ln\left(\frac{c}{4}\right)+C-\frac{1}{2}\right]+2\sum_{n=3}^{\infty}{}'\frac{n i^{n+1}F_n^\ell}{(n^2-1)}}{E_\ell}-\frac{F_1^\ell\sum_{n=1}^{\infty}{}' i^{n+1}\left[F_n^{\ell\prime}+F_n^\ell\,\sigma\left(\frac{n-}{2}\right.\right.}{(E'_\ell)^2}$$

See Note (A) at end of section on Elliptic Cylinder Functions for evaluation of C, σ (n), and $F_n^{\ell\prime}$.

3. Joining Factors.

$$i \, So_\ell(c,z) = \sqrt{2\pi}\, \lambda_\ell' \, Jo_\ell(c,z)$$

$$\lambda_\ell' = \frac{4}{\pi c^2} \sum_{n=2}^{\infty}{}' \, n i^{n-\ell} \frac{F_n^\ell}{F_2^\ell} \qquad \ell = \text{even}$$

$$\lambda_\ell' = \frac{2}{\pi c} \sum_{n=1}^{\infty}{}' \, i^{n-\ell} \frac{F_n^\ell}{F_1^\ell} \qquad \ell = \text{odd}$$

4. Wronskian and Special Values of the Radial Functions.

$$Jo_\ell(c, \cosh\psi) \frac{d}{d\psi} No_\ell(c, \cosh\psi) - No_\ell(c, \cosh\psi) \frac{d}{d\psi} Jo_\ell(c, \cosh\psi) = 1$$

At $z = \cosh\psi \longrightarrow 1$:

$$Jo_\ell(c, \cosh\psi) \longrightarrow 0$$

$$\frac{d}{d\psi} Jo_\ell(c, \cosh\psi) \longrightarrow \frac{1}{\sqrt{2\pi}\, \lambda_\ell'}$$

$$No_\ell(c, \cosh\psi) \longrightarrow -\sqrt{2\pi}\, \lambda_\ell'$$

$$\frac{d}{d\psi} No_\ell(c, \cosh\psi) \longrightarrow \sqrt{2\pi}\, \mu_\ell'$$

The quantity μ_ℓ' is evaluated by differentiating the second forms given above for the No_ℓ solutions and then letting $z=1$.

Note (A)

$C = 0.577215665$ (Jahnke and Emde)

$\sigma(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\frac{1}{n}$; $\sigma(o) = 0$

To obtain $D_n^{\ell'}$ and $F_n^{\ell'}$ (the superscript, ℓ, is omitted for the sake of simplicity):

let $$\gamma_n = \frac{D_n}{D_{n-2}} \quad \text{or} \quad \frac{F_n}{F_{n-2}}$$

then $$\gamma_n = \frac{\frac{c^2}{4}(1 + \delta_{n,2})}{\epsilon - n^2 - \frac{c^2}{4}\gamma_{n+2}}$$

where $$\epsilon = b_\ell - \frac{c^2}{2}$$

$$\delta_{n,2} = \begin{cases} 0 , & n \neq 2 \\ 1 , & n = 2 \end{cases}$$

$$\gamma'_n = \frac{4}{c^2(1+\delta_{n,2})}\gamma_n^2 \left\{ 2n + \frac{\gamma_{n+2}^2}{(1+\delta_{n+2,2})}\left[2(n+2) + \gamma_{n+4}^2 \left(2 \left[n+4 \right] + \cdot \right.\right.\right.$$

$$\frac{\gamma'_n}{\gamma_n} = \frac{D'_n}{D_n} - \frac{D'_{n-2}}{D_{n-2}} \quad \text{or} \quad \frac{\gamma'_n}{\gamma_n} = \frac{F'_n}{F_n} - \frac{F'_{n-2}}{F_{n-2}}$$

ℓ = even: $$D'_0 = 0\ ,\ D'_2 = D_0\gamma'_2$$

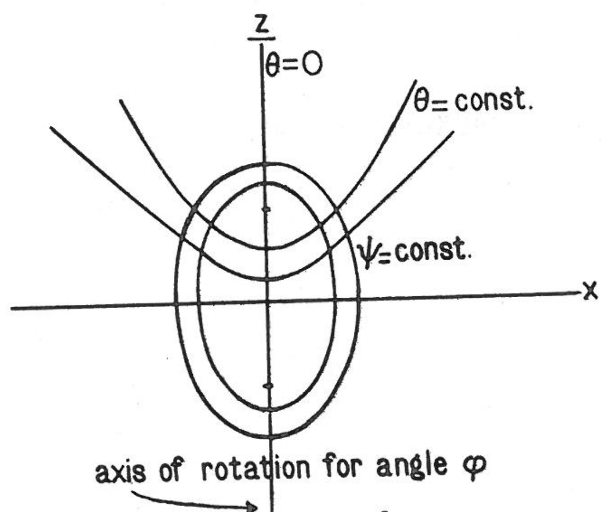
ℓ = odd: $$\gamma_1 = 1\ ,\ D'_1 = \frac{1}{2} D_1\gamma'_1$$

ℓ = even: $$F'_0 = -\,F_2\,\frac{2}{c^2}\left\{4+\gamma_4^2\left[8+\gamma_6^2\ (12+\ldots\ldots)\right]\right\}$$

$$F'_2 = \frac{4\epsilon}{c^2}\,F'_0$$

ℓ = odd: $$\gamma_1 = -1\ ;\ F'_1 = -\,\frac{1}{2}\,F_1\gamma'_1$$

PROLATE SPHEROIDAL FUNCTIONS



Coordinates

$x = \frac{d}{2}\sinh\psi\ \sin\theta\ \cos\varphi$

$y = \frac{d}{2}\sinh\psi\ \sin\theta\ \sin\varphi$

$z = \frac{d}{2}\cosh\psi\ \cos\theta$

Let: $\eta = \cos\theta$, $\xi = \cosh\psi$, $c = \frac{\pi d}{\lambda}$

d= distance between foci.

Wave Equation Yields:

$$\frac{d}{d\eta}(\eta^2-1)\frac{dS}{d\eta} + \left[A+c^2\eta^2 - \frac{m^2}{\eta^2-1}\right] S = 0 \qquad \text{(K)}$$

$$\frac{d}{d\xi}(\xi^2-1)\frac{dR}{d\xi} + \left[A+c^2\xi^2 - \frac{m^2}{\xi^2-1}\right] R = 0 \qquad \text{(L)}$$

Let $z = \eta = \xi$

and $S \text{ or } R = (z^2-1)^{\frac{m}{2}}\ w$

then

$$(1-z^2)w'' - (2m+1)zw' + \left[-A-m(m+1)-c^2z^2\right] w = 0$$

where $-A-m(m+1) \equiv b$ in the paper of Chu and Stratton.

Angular Solution.

$$S_{m\ell}^{(1)}(c, \cos\theta) = {\sum_n}' d_n^\ell P_{m+n}^m (\cos\theta)$$

$$\sum_{n=0}^{\infty}{}' i^{n-\ell} d_n^\ell \frac{\left(\frac{n+2m-1}{2}\right)!}{\left(\frac{n}{2}\right)!} = \frac{\left(\frac{\ell+2m-1}{2}\right)!}{\left(\frac{\ell}{2}\right)!} \qquad \ell = \text{even}$$

$$\sum_{n=1}^{\infty}{}' i^{n-\ell} d_n^\ell \frac{\left(\frac{n+2m}{2}\right)!}{\left(\frac{n-1}{2}\right)!} = \frac{\left(\frac{\ell+2m}{2}\right)!}{\left(\frac{\ell-1}{2}\right)!} \qquad \ell = \text{odd}$$

$$S^{(1)}_{m\ell}(c,0) = P^m_{m+\ell}(0) \; , \; \left.\frac{dS^{(1)}_{m\ell}(c,\eta)}{d\eta}\right|_{\eta=0} = \left.\frac{dP^m_{m+n}(\eta)}{d\eta}\right|_{\eta=0}$$

$$S^{(1)}_{m\ell}(0,\eta) = P^m_{m+\ell}(\eta)$$

Primed summations are over even values of n if ℓ is even, over odd values of n if ℓ is odd.

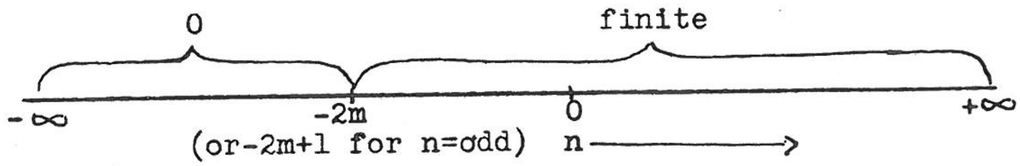
$$\int_{-1}^{1} S^{(1)}_{m\ell}(c,\eta)S^{(1)}_{mk}(c,\eta)\,d\eta = \begin{cases} 0 & , \ \ell \neq k \\ 2\sum_{n=0,1}^{\infty}{}' \dfrac{(d^{\ell}_{n})^2 (n+2m)!}{n!(2n+2m+1)}, & \ell = k \end{cases}$$

The summation of the angular function is from n equals 0 or 1 to n equals ∞, as may be seen by the following schematic representations, using the notations of Chu and Stratton.

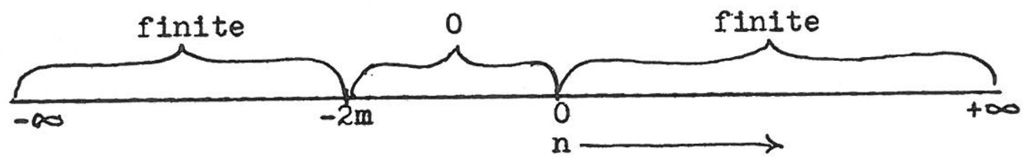
$$S^{(1)}_{m\ell}(c,\eta) = (1-\eta^2)^{\frac{m}{2}} \sum_{n}{}' d^{\ell}_{n} T^{m}_{n}(\eta)$$

where $$P^m_{m+n}(\eta) = (1-\eta^2)^{\frac{m}{2}} T^{m}_{n}(\eta)$$

$$d_n^{\ell}\ (n=\text{even})$$



$$T_n^m$$



Radial Solutions

For $z=\xi$

$$R^{(1)}_{m\ell}(c,z)=\frac{(z^2-1)^{\frac{m}{2}}}{z^m \sum\limits_{n=0,1}^{\infty}{}' d_n^{\ell}\,\frac{(n+2m)!}{n!}} \sum_n{}' i^{\ell-n} d_n^{\ell}\,\frac{(n+2m)!}{n!}\, j_{m+n}(cz)$$

$$\xrightarrow[cz\to\infty]{} \frac{1}{cz}\sin\left(cz-\frac{\ell+m}{2}\pi\right)$$

where $\quad j_{n+m}(cz)=\sqrt{\dfrac{\pi}{2cz}}\; J_{n+m+\frac{1}{2}}(cz)$ $\qquad$ (Spherical Bessel Function)

$$R^{(2)}_{m\ell}(c,z) = \frac{(z^2-1)^{\frac{m}{2}}}{z^m \sum\limits_{n=0,1}^{\infty}{}' d^{\ell}_n \dfrac{(n+2m)!}{n!}} \sum_n{}' \; i^{\ell-n} d^{\ell}_n \frac{(n+2m)!}{n!}\, n_{m+n}(cz)$$

$$\xrightarrow[cz\to\infty]{} -\frac{1}{cz}\cos\left(cz-\frac{\ell+m}{2}\pi\right)$$

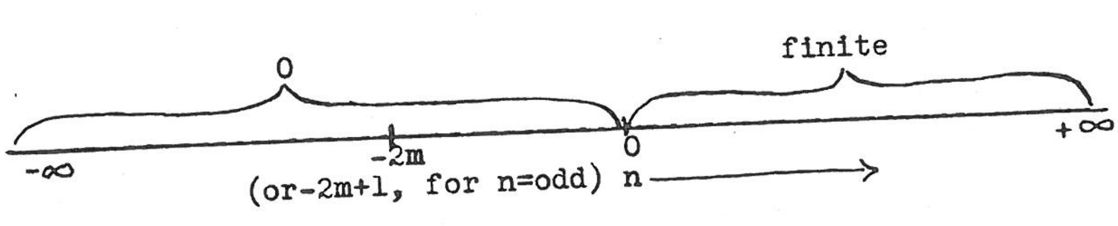
where

$$n_{m+n}(cz) = \sqrt{\frac{\pi}{2cz}}\, N_{n+m+\frac{1}{2}}(cz) = (-1)^{n+m+1}\sqrt{\frac{\pi}{2cz}}\, J_{-n-m-\frac{1}{2}}(cz)$$

The summation of these radial solutions are from n= 0 or 1 to $n=\infty$, as can be seen from the following diagram: In the notation of Chu and Stratton

$$i^{\ell-n} d^{\ell}_n \frac{(n+2m)!}{n!} = a^{\ell}_n$$

$\underline{a^{\ell}_n}$



When the above formula for $R^{(2)}_{m\ell}$ does not converge (namely, z=1 or close to it), or converges too slowly (namely, m= large, such as 3), the following formulas can be used:

<u>ℓ = even:</u>

$$R^{(2)}_{m\ell}(c,z) = \frac{2c^{m-1}\left(\frac{\ell+2m-1}{2}\right)!}{\left(\frac{\ell}{2}\right)!\left(m-\frac{3}{2}\right)!\; d^{\ell}_{-2m}\sum\limits_{n=0}^{\infty}{}' d^{\ell}_{n}\frac{(n+2m)!}{n!}}\sum_{n}{}' d^{\ell}_{n}Q^{m}_{m+n}(z$$

<u>ℓ = odd:</u>

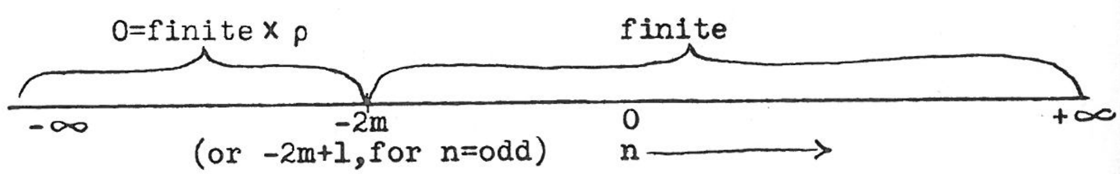
$$R^{(2)}_{m\ell}(c,z) = \frac{-8c^{m-2}\left(\frac{\ell+2m}{2}\right)!}{d^{\ell}_{1-2m}\left(m-\frac{5}{2}\right)!\left(\frac{\ell-1}{2}\right)!\sum\limits_{n=1}^{\infty}{}'\frac{(n+2m)!}{n!}d^{\ell}_{n}}\sum_{n}{}' d^{\ell}_{n}Q^{m}_{m+n}($$

In these cases the summation extends from $-\infty$ to $+\infty$.

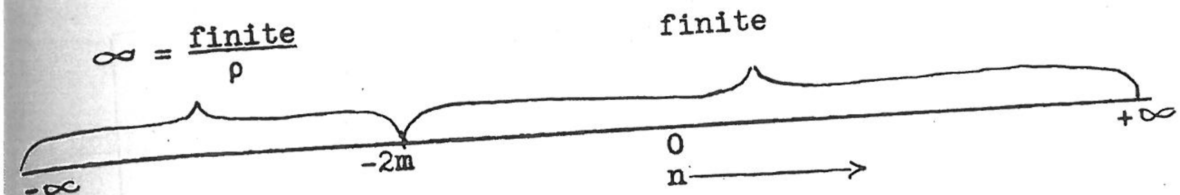
$$Q^{m}_{m+n}(z)=(z^2-1)^{\frac{m}{2}}\,\mathcal{T}^{m}_{n}(z)\ .$$

ρ is a quantity which approaches 0, as n approaches integer.
(See Chu and Stratton)

<u>d^{ℓ}_{n}</u>



$\underline{T^m_n}$



Joining Factors.

$$S^{(1)}_{m\ell}(c,z) = \lambda_{m\ell}\, R^{(1)}_{m\ell}(c,z)$$

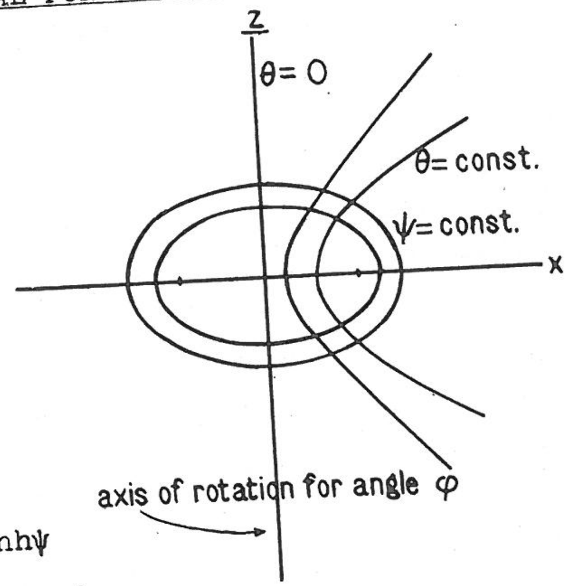
$$\lambda_{m\ell} = \frac{2^{2m+1}\,(m+\frac{1}{2})!\,(\frac{\ell+2m-1}{2})!}{*d^{\ell}_{0}c^{m}(2m)!\,(\frac{\ell}{2})!} \sum_{n=0}^{\infty}{}' \, d^{\ell}_{n}\,\frac{(n+2m)!}{n!} \qquad \ell = \text{even}$$

$$\lambda_{m\ell} = \frac{2^{2m+3}\,(m+\frac{3}{2})!\,(\frac{\ell+2m}{2})!}{*c^{m+1}d^{\ell}_{1}(2m+1)!\,(\frac{\ell-1}{2})!} \sum_{n=1}^{\infty}{}' \, d^{\ell}_{n}\,\frac{(n+2m)!}{n!} \qquad \ell = \text{odd}$$

At $z = 1$:

	$S_{m\ell}^{(1)}(c,1)$	$R_{m\ell}^{(1)}(c,1)$	$\frac{d}{d\psi}R_{m\ell}^{(1)}(c,z)$	$R_{m\ell}^{(2)}(c,1)$	$\frac{d}{d\psi}R_{m\ell}^{(2)}(c,z)$
$m=0$: ℓ=even	$\sum_{n=0}^{\infty}{}' d_n^{\ell}$	$\frac{\sqrt{\pi}\, d_0^{\ell}(\frac{\ell}{2})!}{(\frac{\ell-1}{2})!}$	0	∞	∞
ℓ=odd	$\sum_{n=1}^{\infty}{}' d_n^{\ell}$	$\frac{\sqrt{\pi}\, c d_1^{\ell}(\frac{\ell-1}{2})!}{6(\frac{\ell}{2})!}$	0	∞	∞
$m=1$: ℓ=even	0	0	$\sqrt{\pi}\, \frac{c d_0^{\ell}(\frac{\ell}{2})!}{6(\frac{\ell+1}{2})!}$	∞	∞
ℓ=odd	0	0	$\frac{\sqrt{\pi} c^2 d_1^{\ell}(\frac{\ell-1}{2})!}{20(\frac{\ell+2}{2})!}$	∞	∞
$m > 1$	0	0	0	∞	∞

OBLATE SPHEROIDAL FUNCTIONS



Coordinates:

$x = \frac{d}{2} \cosh\psi \sin\theta \cos\varphi$

$y = \frac{d}{2} \cosh\psi \sin\theta \sin\varphi$

$z = \frac{d}{2} \sinh\psi \cos\theta$

Let: $\eta = \cos\theta$; $\zeta = \sinh\psi$

d= distance between foci; $c = \frac{\pi d}{\lambda}$

The Wave Equation Yields:

$$\frac{d}{d\eta}(\eta^2 - 1)\frac{dS}{d\eta} + \left[B - c^2\eta^2 - \frac{m^2}{\eta^2 - 1}\right] S = 0 \qquad \text{(M)}$$

$$\frac{d}{d\zeta}(\zeta^2 + 1)\frac{dR}{d\zeta} + \left[B + c^2\zeta^2 + \frac{m^2}{\zeta^2 + 1}\right] R = 0 \qquad \text{(N)}$$

If c is replaced by +ic, equation (M) becomes of the same form as equation (K), and hence will have the same solutions, provided c (prolate) is replaced by -ic (oblate) in the solutions.

If c is replaced by +ic, and ζ by $-i\xi$, equation (N) will be identical with equation (L), and will also have the same solutions, provided that c (prolate) is replaced by -ic (oblate), and ξ is replaced by $i\zeta$ in the solutions.

For example, a solution to equation (M) is:

$$S_{m\ell}^{(1)}(-ic,\cos\theta) = \sum_{n=0,1}^{\infty}{}' f_n^{\ell} P_{m+n}^{m}(\cos\theta)$$

where the coefficients f_n^{ℓ} are, of course, different in value from the coefficients d_n^{ℓ} of the prolate case, but are obtained in the same way.

All the prolate equations for $S_{m\ell}^{(1)}$, $R_{m\ell}^{(1)}$, and $R_{m\ell}^{(2)}$ apply for the oblate case provided that:

1. c(prolate) be replaced by -ic(oblate),
2. $\cos\theta = \eta$ remains the same,
3. the coefficient d_n^{ℓ} be replaced by f_n^{ℓ},
4. $z=\xi = \cosh\psi$ for the prolate case be replaced by $z= i\zeta = i \sinh\psi$ for the oblate case,
5. the positive square root is to be chosen in all cases .

SPECIAL VALUES FOR THE OBLATE SPHEROIDAL FUNCTIONS

At $\zeta = 0$:

$$S_{m\ell}^{(1)}(-ic,0) = \frac{i^{\ell+m} 2^m \left(\frac{\ell+2m-1}{2}\right)!}{\sqrt{\pi}\ \left(\frac{\ell}{2}\right)!} \qquad \ell = \text{even}$$

$$=0 \qquad \ell = \text{odd}$$

$$R_{m\ell}^{(1)}(-ic,i0) = \frac{i^{\ell}\sqrt{\pi}\ c^m f_0^{\ell}(2m)!}{2^{m+1}(m+\frac{1}{2})!\sum_{n=0}^{\infty}{}' f_n^{\ell}\ \frac{(n+2m)!}{n!}} \qquad \ell = \text{even}$$

$$=0 \qquad \ell = \text{odd}$$

$$\frac{d}{d\psi} R^{(1)}_{m\ell}(-ic,i\zeta)\Big|_{\zeta=0} = 0 \qquad \ell = \text{even}$$

$$= \frac{i^{\ell-1}\sqrt{\pi}\; c^{m+1} f^{\ell}_{1}(2m+1)!}{2^{m+2}(m+\frac{3}{2})!\sum\limits_{n=1}^{\infty}{}' f^{\ell}_{n}\frac{(n+2m)!}{n!}} \qquad \ell = \text{odd}$$

$$R^{(2)}_{m\ell}(-ic,i0) = \frac{i^{\ell}\sqrt{\pi}\; 2^{m} c^{m-1}}{(m-\frac{3}{2})!\; f^{\ell}_{-2m}\sum\limits_{n=0}^{\infty}{}' f^{\ell}_{n}\frac{(n+2m)!}{n!}}\left[\frac{(\frac{\ell+2m-1}{2})!}{(\frac{\ell}{2})!}\right]^{2}$$

ℓ = even

$$R^{(2)}_{m\ell}(-ic,i0) = \frac{2^{m+3}c^{m-2}(\frac{\ell+2m}{2})!}{\sqrt{\pi}\; f^{\ell}_{1-2m}(m-\frac{5}{2})!(\frac{\ell-1}{2})!\sum\limits_{n=1}^{\infty}{}' f^{\ell}_{n}\frac{(n+2m)!}{n!}} \times$$

$$\left[\frac{\pi}{2}\sum_{n=1}^{\infty}{}' f^{\ell}_{n} i^{n+1}\frac{(\frac{n+2m-1}{2})!}{(\frac{n}{2})!} + \frac{1}{2}\sum_{n=1}^{2m-1}{}' f^{\ell}_{-n}(\frac{2m-n-1}{2})!(\frac{n-2}{2})!\right.$$

$$\left. + (-1)^{m}\sum_{n=2m+1}^{\infty}{}' i^{n-1}\frac{f^{\ell}_{-n}}{\rho}\frac{(\frac{n-2}{2})!}{(\frac{n-2m-1}{2})!}\right] \qquad \ell = \text{odd}$$

$$\frac{d}{d\psi} R^{(2)}_{m\ell}(-ic,i\zeta)\Big|_{\zeta=0} = \frac{-2^{m+1}c^{m-1}(\frac{\ell+2m-1}{2})!}{\sqrt{\pi}\;(\frac{\ell}{2})!(m-\frac{3}{2})! f^{\ell}_{-2m}\sum\limits_{n=0}^{\infty}{}' f^{\ell}_{n}\frac{(n+2m)!}{n!}} \times$$

$$\left[\pi\sum_{n=0}^{\infty}{}' i^{n} f^{\ell}_{n}\frac{(\frac{n+2m}{2})!}{(\frac{n-1}{2})!} + \sum_{n=2}^{2m}{}' f^{\ell}_{-n}(\frac{2m-n}{2})!(\frac{n-1}{2})!\right.$$

$$\left. +(-1)^{m+1}2\sum_{n=2m+2}^{\infty}{}' i^{n}\frac{f^{\ell}_{-n}}{\rho}\frac{(\frac{n-1}{2})!}{(\frac{n-2m-2}{2})!}\right] \qquad \ell = \text{even}$$

$$\frac{d}{d\psi} R^{(2)}_{m\ell}(-ic,i\zeta)\Big|_{\zeta=0} = \frac{i^{\ell-1}2^{m+3}\sqrt{\pi}\; c^{m-2}}{f^{\ell}_{1-2m}(m-\frac{5}{2})!\sum\limits_{n=1}^{\infty}{}' f^{\ell}_{n}\frac{(n+2m)!}{n!}}\left[\frac{(\frac{\ell+2m}{2})!}{(\frac{\ell-1}{2})!}\right]^{2}$$

ℓ = odd

For certain values of c and A (or c and B) it has been found, both theoretically and as a result of numerical calculations, that one or more of the negative coefficients for the range $-2m \leq n < 0$ for n even, and $-2m+1 \leq n < 1$ for n odd, become infinite or zero. However, in all such cases investigated, it can be shown that each of the solutions $S^{(1)}_{m\ell}$, $S^{(2)}_{m\ell}$ (see Chu and Stratton), $R^{(1)}_{m\ell}$ or $R^{(2)}_{m\ell}$ behave properly. The essential points of the argument are: (1) m is always an integer, (2) we will always observe the rule that $\rho \longrightarrow 0$ before $c \longrightarrow c_k$; where ρ is a quantity which approaches zero as n approaches a whole number (see Chu and Stratton), and c_k is the value of c for which one or more negative coefficients become infinite.

NOTES

In addition to the formulas to be found in the first part of the article by Chu and Stratton, the following will be found useful:

(1)
$$Q^m_{m-n}(z) = \frac{P^m_{n-m-1}(z)}{\rho} \qquad \text{for } |n| > 2m$$

(to be used for evaluating part of the negative half of the summation in the second solution for $R^{(2)}_{m\ell}$).

(2) For m = positive integer, and n unrestricted:

$$Q^m_{m+n}(z) = \frac{(-1)^m 2^{m+n}(m+n)!\,(n+2m)!}{(2n+2m+1)!} \, \frac{(z^2-1)^{\frac{m}{2}}}{z^{n+2m+1}} \times$$

$$F\left(\frac{n+2m+2}{2}, \frac{n+2m+1}{2}; m+n+\frac{3}{2}; \frac{1}{z^2}\right)$$

(3) Some auxiliary tables available,

A. Bessel Functions

1. $J_n(x)$

a. Jahnke-Emde, "Tables of Functions"

b. Watson, "Theory of Bessel Functions"

c. Report of the British Association for the Advancement of Science, 1915

2. $N_n(x)$

a. Jahnke-Emde, "Tables of Functions"

b. Watson, "Theory of Bessel Functions"

$$N_n \equiv Y_n$$

c. Report of the British Association for the Advancement of Science, 1914

$$N_n \equiv -\frac{2}{\pi} G_n$$

3. $J_{n+\frac{1}{2}}(x)$ and $J_{-n-\frac{1}{2}}(x)$

a. Jahnke-Emde, "Tables of Functions"

b. Report of the British Association for the Advancement of Science, 1925

c. Report of the British Association for the Advancement of Science, 1914, 1916, 1922

$$J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, S_n(x)$$

$$J_{-n-\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{2}{\pi x}}\, C_n(x)$$

d. W.P.A. Tables to be published soon.

B. Legendre Functions

1. P_n^m (cosθ) and its derivative

a. Jahnke-Emde, "Tables of Functions"

b. Tallquist, Acta Societatis Scientiarum Fennicae, Vol. 33, 1908, No. 9

c. W.P.A. Tables to be published soon

2. P_n^m (x); P_n^m(ix) and their derivatives, for $x > 1$

a. W.P.A. Tables to be published soon

3. Q_n^m (x); Q_n^m(ix) and their derivatives for $x > 1$

a. W.P.A. Tables to be published soon.

R. Albagli Hutner.

TABLES

In the tables of the spheroidal coefficients, the number within the parentheses indicates the number of zeros between the decimal point and the first significant figure. Thus:

$$.(3)24692 \equiv .00024692$$

If the sign is the same throughout the whole length of a column, the sign is indicated at the top of the column; otherwise it is given in front of each number.

Although five significant figures have been published in the tables of coefficients, the fifth figure must be considered doubtful. An exception occurs in the case of the row c=1.4 in the table "Prolate Spheroidal Functions $m=3$, $\ell=0$" for negative subscripts. Here the calculations involve a small difference of two large numbers and more than three significant figures appears to be unwarranted.

A few of the sets of separation constants, $A_{m\ell}$ and $B_{m\ell}$ are listed directly on the pages of the corresponding coefficients;however most of them have been placed together on two separate pages. The subscripts of $A_{2,1}$ denote $m=2$, $\ell=1$.

In some of the tables of coefficients with negative subscripts, the coefficients become infinite for a certain value of c, namely, c_k. When c_k falls between two consecutive tabulated values of c, a row of infinities has been inserted in the table.

c	b_0	$+D_0^{(0)}$	$-D_2^{(0)}$	$+D_4^{(0)}$	$-D_6^{(0)}$	$+D_8^{(0)}$	$-D_{10}^{(0)}$	$+D_{12}^{(0)}$
0	0	1.0000	0	0	0	0	0	0
0.1	0.00500	1.0013	0.00125	0	0	0	0	0
0.2	.01995	1.0050	.00503	0	0	0	0	0
0.4	.07920	1.0204	.02041	0.00005	0	0	0	0
0.5	.12304	1.0323	.03237	.00012	0	0	0	0
0.6	.17600	1.0468	.04708	.00026	0	0	0	0
0.8	.30723	1.0858	.08662	.00087	0	0	0	0
1.0	0.46896	1.1393	0.14145	0.00221	0.00002	0	0	0
1.2	.65609	1.2100	.21481	.00482	.00005	0	0	0
1.4	.86294	1.3016	.31095	.00945	.00013	0	0	0
1.5	.97190	1.3566	.36923	.01286	.00020	0	0	0
1.6	1.0836	1.4183	.43526	.01722	.00030	0	0	0
1.8	1.3125	1.5654	.59428	.02955	.00066	0.00001	0	0
2.0	1.5449	1.7488	0.79595	0.04845	0.00133	0.00002	0	0
2.2	1.7771	1.9757	1.0498	.07650	.00253	.00004	0	0
2.4	2.0069	2.2543	1.3669	.11704	.00458	.00011	0	0
2.5	2.1204	2.4160	1.5533	.14327	.00606	.00014	0	0
2.6	2.2329	2.5942	1.7608	.17433	.00794	.00021	0	0
2.8	2.4548	3.0070	2.2478	.25375	.01330	.00040	0.00001	0
3.0	2.6728	3.5061	2.8474	0.36209	0.02158	0.00074	0.00002	0
3.2	2.8871	4.1082	3.5832	.50789	.03410	.00132	.00003	0
3.4	3.0984	4.8328	4.4843	.70185	.05261	.00228	.00006	0
3.5	3.2031	5.2485	5.0075	.82092	.06482	.00299	.00010	0
3.6	3.3073	5.7040	5.5856	.95734	.07950	.00384	.00012	0
3.8	3.5142	6.7505	6.9297	1.2911	.11796	.00630	.00022	0
4.0	3.7195	8.0072	8.5688	1.7240	0.17221	0.01011	0.00039	0.00001
4.2	3.9236	9.5161	10.566	2.2821	.24780	.01590	.00067	.00002
4.4	4.1268	11.328	12.997	2.9977	.35203	.02458	.00113	.00004
4.5	4.2282	12.366	14.405	3.4269	.41774	.03042	.00148	.00005

c	b_1	$+D_1^{(1)}$	$-D_3^{(1)}$	$+D_5^{(1)}$	$-D_7^{(1)}$	$+D_9^{(1)}$	$-D_{11}^{(1)}$	$+D_{13}^{(1)}$
0	1.0000	1.0000	0	0	0	0	0	0
0.1	1.0075	1.0003	0.00030	0	0	0	0	0
0.2	1.0300	1.0013	.00125	0	0	0	0	0
0.4	1.1199	1.0050	.00505	0.00001	0	0	0	0
0.5	1.1870	1.0079	.00793	.00002	0	0	0	0
0.6	1.2690	1.0114	.01151	.00004	0	0	0	0
0.8	1.4767	1.0207	.02080	.00014	0	0	0	0
1.0	1.7419	1.0330	0.03330	0.00035	0	0	0	0
1.2	2.0631	1.0486	.04937	.00076	0.00001	0	0	0
1.4	2.4381	1.0681	.06953	.00145	.00001	0	0	0
1.5	2.6451	1.0794	.08132	.00195	.00002	0	0	0
1.6	2.8646	1.0919	.09447	.00259	.00003	0	0	0
1.8	3.3395	1.1209	.12519	.00436	.00007	0	0	0
2.0	3.8591	1.1560	0.16287	0.00704	0.00015	0	0	0
2.2	4.4190	1.1983	.20900	.01101	.00028	0	0	0
2.4	5.0139	1.2493	.26552	.01675	.00051	0.00001	0	0
2.5	5.3228	1.2786	.29840	.02050	.00069	.00001	0	0
2.6	5.6383	1.3108	.33484	.02495	.00090	.00002	0	0
2.8	6.2857	1.3849	.41994	.03650	.00154	.00004	0	0
3.0	6.9496	1.4744	0.52450	0.05259	0.00255	0.00007	0	0
3.2	7.6235	1.5821	.65290	.07477	.00413	.00014	0	0
3.4	8.3017	1.7117	.81046	.10505	.00654	.00024	0.00001	0
3.5	8.6407	1.7859	.90204	.12398	.00820	.00032	.00001	0
3.6	8.9790	1.8671	1.0033	.14593	.01019	.00043	.00002	0
3.8	9.6519	2.0528	1.2386	.20062	.01560	.00072	.00002	0
4.0	10.318	2.2740	1.5247	0.27304	0.02349	0.00121	0.00005	0
4.2	10.976	2.5365	1.8716	.36803	.03483	.00195	.00008	0
4.4	11.626	2.8468	2.2905	.49156	.05087	.00313	.00011	0
4.5	11.948	3.0224	2.5313	.56625	.06117	.00393	.00018	0.00001

EVEN ELLIPTIC CYLINDER FUNCTIONS $\ell = 2$

c	b_2	$+D_0^{(2)}$	$+D_2^{(2)}$	$-D_4^{(2)}$	$+D_6^{(2)}$	$-D_8^{(2)}$	$+D_{10}^{(2)}$	$-D_{12}^{(2)}$
0	4.0000	0	1.0000	0	0	0	0	0
0.1	4.0050	0.00062	0.99958	0.00020	0	0	0	0
0.2	4.0200	.00250	.99833	.00083	0	0	0	0
0.4	4.0807	.00993	.99331	.00323	0	0	0	0
0.5	4.1266	.01546	.98970	.00516	0.00001	0	0	0
0.6	4.1834	.02215	.98522	.00739	.00002	0	0	0
0.8	4.3306	.03886	.97413	.01303	.00007	0	0	0
1.0	4.5258	0.05963	0.96027	0.02005	0.00014	0	0	0
1.2	4.7731	.08387	.94429	.02846	.00032	0	0	0
1.4	5.0770	.11084	.92675	.03816	.00058	0.00001	0	0
1.5	5.2517	.12507	.91766	.04351	.00077	.00001	0	0
1.6	5.4422	.13970	.90850	.04918	.00099	.00001	0	0
1.8	5.8726	.16959	.89036	.06150	.00157	.00002	0	0
2.0	6.3713	0.19975	0.87320	0.07530	0.00238	0.00004	0	0
2.2	6.9398	.22962	.85773	.09077	.00349	.00007	0	0
2.4	7.5782	.25885	.84453	.10824	.00498	.00012	0	0
2.5	7.9232	.27319	.83891	.11784	.00591	.00016	0	0
2.6	8.2850	.28736	.83401	.12810	.00693	.00020	0	0
2.8	9.0577	.31523	.82632	.15084	.00960	.00032	0.00001	0
3.0	9.8927	0.34283	0.82167	0.17706	0.01306	0.00050	0.00001	0
3.2	10.786	.37056	.82012	.20751	.01758	.00077	.00002	0
3.4	11.732	.39902	.82173	.24309	.02349	.00118	.00003	0
3.5	12.223	.41370	.82373	.26314	.02710	.00144	.00005	0
3.6	12.725	.42883	.82656	.28490	.03121	.00175	.00006	0
3.8	13.760	.46077	.83473	.33428	.04128	.00260	.00010	0
4.0	14.829	0.49576	0.84640	0.39290	0.05440	0.00382	0.00016	0.00001
4.2	15.926	.53481	.86173	.46271	.07147	.00556	.00027	.00001
4.4	17.043	.57913	.88100	.54617	.09367	.00804	.00042	.00002
4.5	17.606	.60370	.89217	.59387	.10713	.00964	.00053	.00002

$\ell=3$

c	b_3	$+D_1^{(3)}$	$+D_3^{(3)}$	$-D_5^{(3)}$	$+D_7^{(3)}$	$-D_9^{(3)}$	$+D_{11}^{(3)}$	$-D_{13}^{(3)}$
0	9.0000	0	1.0000	0	0	0	0	0
0.1	9.0050	.00031	0.99985	0.00016	0	0	0	0
0.2	9.0200	.00125	.99937	.00063	0	0	0	0
0.4	9.0801	.00501	.99748	.00249	0	0	0	0
0.5	9.1253	.00784	.99597	.00382	0	0	0	0
0.6	9.1805	.01131	.99428	.00559	0.00001	0	0	0
0.8	9.3217	.02019	.98967	.00990	.00004	0	0	0
1.0	9.5042	0.03171	0.98357	0.01537	0.00010	0	0	0
1.2	9.7288	.04593	.97585	.02199	.00020	0	0	0
1.4	9.9969	.06291	.96637	.02964	.00036	0	0	0
1.5	10.148	.07245	.96086	.03384	.00048	0	0	0
1.6	10.310	.08271	.95498	.03829	.00061	0.00001	0	0
1.8	10.669	.10535	.94156	.04786	.00097	.00001	0	0
2.0	11.078	0.13082	0.92599	0.05825	0.00146	0.00002	0	0
2.2	11.539	.15905	.90824	.06936	.00211	.00004	0	0
2.4	12.056	.18990	.88835	.08111	.00294	.00006	0	0
2.5	12.337	.20622	.87760	.08719	.00343	.00007	0	0
2.6	12.633	.22311	.86642	.09341	.00398	.00009	0	0
2.8	13.274	.25835	.84277	.10623	.00526	.00014	0	0
3.0	13.983	0.29519	0.81777	0.11957	0.00682	0.00022	0	0
3.2	14.765	.33315	.79194	.13349	.00870	.00031	0.00001	0
3.4	15.624	.37174	.76584	.14812	.01097	.00045	.00001	0
3.5	16.082	.39114	.75289	.15577	.01226	.00053	.00002	0
3.6	16.561	.41054	.74008	.16368	.01368	.00062	.00002	0
3.8	17.577	.44924	.71514	.18048	.01694	.00087	.00003	0
4.0	18.671	0.48769	0.69143	0.19884	0.02088	0.00119	0.00004	0
4.2	19.841	.52593	.66919	.21921	.02564	.00162	.00006	0
4.4	21.084	.56418	.64855	.24207	.03144	.00219	.00010	0
4.5	21.731	.58341	.63881	.25460	.03481	.00255	.00012	0

EVEN ELLIPTIC CYLINDER FUNCTIONS $\ell=4$

c	b_4	$+D_0^{(4)}$	$+D_2^{(4)}$	$+D_4^{(4)}$	$-D_6^{(4)}$	$+D_8^{(4)}$	$-D_{10}^{(4)}$	$+D_{12}^{(4)}$
0	16.000	0	0	1.0000	0	0	0	0
0.1	16.005	0	0.00021	0.99992	0.00012	0	0	0
0.2	16.020	0	.00083	.99967	.00050	0	0	0
0.4	16.080	0.00001	.00333	.99866	.00200	0	0	0
0.5	16.125	.00002	.00526	.99791	.00319	0	0	0
0.6	16.180	.00004	.00748	.99696	.00449	0.00001	0	0
0.8	16.321	.00013	.01327	.99453	.00796	.00003	0	0
1.0	16.502	0.00033	0.02066	0.99136	0.01241	0.00007	0	0
1.2	16.724	.00067	.02965	.98733	.01778	.00013	0	0
1.4	16.988	.00123	.04020	.98241	.02408	.00025	0	0
1.5	17.136	.00162	.04603	.97960	.02757	.00032	0	0
1.6	17.294	.00209	.05224	.97652	.03128	.00042	0	0
1.8	17.642	.00333	.06578	.96957	.03934	.00067	0.00001	0
2.0	18.034	0.00504	0.08073	0.96144	0.04821	0.00101	0.00001	0
2.2	18.470	.00732	.09707	.95201	.05783	.00146	.00002	0
2.4	18.951	.01028	.11472	.94111	.06815	.00205	.00004	0
2.5	19.209	.01204	.12403	.93513	.07356	.00240	.00005	0
2.6	19.479	.01403	.13364	.92869	.07911	.00280	.00006	0
2.8	20.055	.01868	.15375	.91455	.09060	.00371	.00008	0
3.0	20.681	0.02432	0.17496	0.89857	0.10256	0.00483	0.00013	0
3.2	21.359	.03109	.19720	.88062	.11489	.00616	.00018	0
3.4	22.092	.03904	.22034	.86061	.12746	.00774	.00027	0.00001
3.5	22.480	.04348	.23221	.84981	.13382	.00863	.00031	.00001
3.6	22.882	.04825	.24426	.83849	.14020	.00957	.00037	.00001
3.8	23.733	.05876	.26881	.81425	.15301	.01167	.00050	.00002
4.0	24.650	0.07059	0.29381	0.78799	0.16580	0.01407	0.00068	0.00002
4.2	25.636	.08368	.31907	.75987	.17853	.01677	.00089	.00003
4.4	26.695	.09792	.34424	.72987	.19112	.02022	.00118	.00004
4.5	27.254	.10551	.35699	.71482	.19753	.02148	.00132	.00005

ODD ELLIPTIC CYLINDER FUNCTIONS $\ell=1$

c	b_1'	$+F_1^{(1)}$	$-F_3^{(1)}$	$+F_5^{(1)}$	$-F_7^{(1)}$	$+F_9^{(1)}$	$-F_{11}^{(1)}$	$+F_{13}^{(1)}$
0	1.0000	1.0000	0	0	0	0	0	0
0.1	1.0025	1.0009	0.00031	0	0	0	0	0
0.2	1.0099	1.0038	.00125	0	0	0	0	0
0.4	1.0398	1.0152	.00506	0.00001	0	0	0	0
0.5	1.0595	1.0235	.00786	.00002	0	0	0	0
0.6	1.0890	1.0343	.01150	.00004	0	0	0	0
0.8	1.1569	1.0618	.02081	.00014	0	0	0	0
1.0	1.2424	1.0980	0.03325	0.00034	0	0	0	0
1.2	1.3445	1.1440	.04920	.00073	0	0	0	0
1.4	1.4618	1.2006	.06914	.00138	0.00001	0	0	0
1.5	1.5257	1.2333	.08079	.00185	.00002	0	0	0
1.6	1.5928	1.2691	.09368	.00243	.00003	0	0	0
1.8	1.7359	1.3510	.12354	.00402	.00007	0	0	0
2.0	1.8898	1.4480	0.15964	0.00636	0.00013	0	0	0
2.2	2.0527	1.5623	.20307	.00970	.00024	0	0	0
2.4	2.2235	1.6965	.25512	.01434	.00042	0	0	0
2.5	2.3113	1.7719	.28487	.01728	.00054	0.00001	0	0
2.6	2.4006	1.8535	.31739	.02069	.00070	.00001	0	0
2.8	2.5830	2.0367	.39173	.02924	.00114	.00003	0	0
3.0	2.7697	2.2504	0.48040	0.04059	0.00180	0.00005	0	0
3.2	2.9597	2.4994	.58605	.05550	.00278	.00009	0	0
3.4	3.1524	2.7892	.71186	.07490	.00420	.00015	0	0
3.5	3.2495	2.9515	.78350	.08664	.00513	.00019	0	0
3.6	3.3471	3.1267	.86165	.09995	.00623	.00024	0	0
3.8	3.5434	3.5197	1.0399	.13205	.00908	.00039	0.00001	0
4.0	3.7408	3.9774	1.2521	0.17295	0.01304	0.00061	0.00002	0
4.2	3.9391	4.5107	1.5045	.22477	.01847	.00095	.00003	0
4.4	4.1381	5.1327	1.8049	.29012	.02587	.00145	.00006	0
4.5	4.2377	5.4815	1.9758	.32884	.03050	.00178	.00007	0

$\ell=2$

c	b'_2	$+F_2^{(2)}$	$-F_4^{(2)}$	$+F_6^{(2)}$	$-F_8^{(2)}$	$+F_{10}^{(2)}$	$-F_{12}^{(2)}$
0	4.0000	0.50000	0	0	0	0	0
0.1	4.0050	.50021	0.00010	0	0	0	0
0.2	4.0200	.50083	.00042	0	0	0	0
0.4	4.0799	.50335	.00168	0	0	0	0
0.5	4.1247	.50525	.00263	0.00001	0	0	0
0.6	4.1793	.50758	.00381	.00001	0	0	0
0.8	4.3179	.51359	.00685	.00003	0	0	0
1.0	4.4948	0.52147	0.01086	0.00008	0	0	0
1.2	4.7092	.53133	.01593	.00018	0	0	0
1.4	4.9600	.54332	.02217	.00034	0	0	0
1.5	5.0987	.55018	.02575	.00045	0	0	0
1.6	5.2459	.55761	.02968	.00059	0.00001	0	0
1.8	5.5655	.57445	.03867	.00098	.00001	0	0
2.0	5.9170	0.59407	0.04930	0.00154	0.00002	0	0
2.2	6.2988	.61681	.06180	.00233	.00005	0	0
2.4	6.7087	.64303	.07649	.00343	.00008	0	0
2.5	6.9237	.65753	.08473	.00412	.00011	0	0
2.6	7.1449	.67311	.09366	.00491	.00013	0	0
2.8	7.6050	.70757	.11371	.00691	.00022	0	0
3.0	8.0870	0.74702	0.13713	0.00954	0.00036	0.00001	0
3.2	8.5885	.79211	.16447	.01299	.00055	.00001	0
3.4	9.1073	.84358	.19635	.01745	.00084	.00003	0
3.5	9.3726	.87202	.21425	.02013	.00102	.00003	0
3.6	9.6414	.90236	.23357	.02316	.00124	.00005	0
3.8	10.189	.96948	.27701	.03047	.00180	.00007	0
4.0	10.747	1.0461	0.32773	0.03974	0.00260	0.00010	0
4.2	11.315	1.1337	.38698	.05142	.00370	.00017	0.00001
4.4	11.890	1.2337	.45621	.06609	.00519	.00026	.00001
4.5	12.180	1.2888	.49503	.07475	.00613	.00032	.00001

c	b_3'	$+F_1^{(3)}$	$+F_3^{(3)}$	$-F_5^{(3)}$	$+F_7^{(3)}$	$-F_9^{(3)}$	$+F_{11}^{(3)}$
0	9.0000	0	0.33333	0	0	0	0
0.1	9.0050	0.00010	.33339	0.00005	0	0	0
0.2	9.0200	.00042	.33354	.00021	0	0	0
0.4	9.0801	.00166	.33417	.00084	0	0	0
0.5	9.1252	.00259	.33461	.00128	0	0	0
0.6	9.1805	.00373	.33522	.00189	0	0	0
0.8	9.3215	.00660	.33671	.00337	0.00001	0	0
1.0	9.5037	0.01026	0.33866	0.00530	0.00003	0	0
1.2	9.7274	.01467	.34109	.00768	.00007	0	0
1.4	9.9932	.01982	.34401	.01055	.00013	0	0
1.5	10.142	.02266	.34567	.01217	.00017	0	0
1.6	10.302	.02567	.34747	.01393	.00022	0	0
1.8	10.653	.03220	.35152	.01785	.00036	0	0
2.0	11.048	0.03937	0.35620	0.02237	0.00056	0.00001	0
2.2	11.486	.04717	.36158	.02752	.00084	.00001	0
2.4	11.967	.05558	.36771	.03338	.00121	.00003	0
2.5	12.224	.06001	.37107	.03660	.00144	.00003	0
2.6	12.491	.06460	.37469	.04004	.00170	.00004	0
2.8	13.057	.07426	.38257	.04756	.00234	.00007	0
3.0	13.664	0.08459	0.39149	0.05607	0.00317	0.00010	0
3.2	14.311	.09562	.40153	.06571	.00424	.00015	0
3.4	14.996	.10743	.41284	.07661	.00558	.00022	0.00001
3.5	15.352	.11366	.41900	.08259	.00638	.00027	.00001
3.6	15.717	.12013	.42554	.08897	.00728	.00033	.00001
3.8	16.473	.13383	.43980	.10298	.00940	.00047	.00002
4.0	17.261	0.14869	0.45579	0.11890	0.01204	0.00067	0.00002
4.2	18.080	.16488	.47369	.13700	.01531	.00094	.00004
4.4	18.925	.18263	.49375	.15763	.01935	.00131	.00006
4.5	19.357	.19216	.50464	.16900	.02171	.00154	.00007

ODD ELLIPTIC CYLINDER FUNCTIONS

$l=4$

c	b'_4	$+F_2^{(4)}$	$+F_4^{(4)}$	$-F_6^{(4)}$	$+F_8^{(4)}$	$-F_{10}^{(4)}$	$+F_{12}^{(4)}$
0	16.000	0	0.25000	0	0	0	0
0.1	16.005	0.00005	.25002	0.00003	0	0	0
0.2	16.020	.00021	.25008	.00013	0	0	0
0.4	16.080	.00083	.25033	.00050	0	0	0
0.5	16.125	.00130	.25052	.00078	0	0	0
0.6	16.180	.00188	.25075	.00113	0	0	0
0.8	16.321	.00335	.25133	.00201	0.00001	0	0
1.0	16.502	0.00525	0.25207	0.00315	0.00002	0	0
1.2	16.724	.00759	.25297	.00455	.00004	0	0
1.4	16.988	.01037	.25403	.00623	.00006	0	0
1.5	17.136	.01193	.25462	.00717	.00008	0	0
1.6	17.294	.01360	.25525	.00818	.00011	0	0
1.8	17.642	.01729	.25662	.01041	.00018	0	0
2.0	18.033	0.02145	0.25815	0.01294	0.00027	0	0
2.2	18.468	.02610	.25984	.01578	.00040	0.00001	0
2.4	18.948	.03123	.26169	.01895	.00057	.00001	0
2.5	19.204	.03398	.26267	.02066	.00068	.00001	0
2.6	19.472	.03686	.26371	.02245	.00079	.00002	0
2.8	20.043	.04299	.26590	.02633	.00108	.00002	0
3.0	20.660	0.04964	0.26827	0.03059	0.00144	0.00004	0
3.2	21.324	.05682	.27085	.03527	.00189	.00006	0
3.4	22.035	.06453	.27365	.04041	.00245	.00008	0
3.5	22.408	.06860	.27514	.04317	.00278	.00010	0
3.6	22.793	.07281	.27670	.04605	.00314	.00012	0
3.8	23.599	.08166	.28001	.05225	.00397	.00017	0.00001
4.0	24.452	0.09111	0.28362	0.05906	0.00499	0.00024	0.00001
4.2	25.351	.10120	.28758	.06655	.00621	.00033	.00001
4.4	26.296	.11199	.29191	.07481	.00761	.00045	.00002
4.5	26.786	.11765	.29424	.07926	.00853	.00052	.00002

PROLATE SEPARATION CONSTANTS

c	$A_{0,0}$	$A_{0,1}$	$A_{0,2}$	$A_{0,3}$	$A_{1,0}$	$A_{1,1}$	$A_{1,2}$	$A_{3,0}$
0	0	-2.00000	-6.00000	-12.00000	-2.00000	-6.00000	-12.00000	-12.00000
0.1	-0.00333	-2.00600	-6.00523	-12.00511	-2.00200	-6.00429	-12.00467	-12.00111
0.2	-0.01331	-2.02399	-6.02096	-12.02045	-2.00799	-6.01714	-12.01867	-12.00444
0.4	-0.05296	-2.09582	-6.08407	-12.08186	-2.03188	-6.06847	-12.07470	-12.01775
0.5	-0.08242	-2.14957	-6.13158	-12.12798	-2.04972	-6.10690	-12.11675	-12.02772
0.6	-0.11810	-2.21511	-6.18986	-12.18443	-2.07141	-6.15378	-12.16817	-12.03987
0.8	-0.20739	-2.38118	-6.33927	-12.32848	-2.12616	-6.27270	-12.29920	-12.07071
1.0	-0.31900	-2.59308	-6.53347	-12.51446	-2.19555	-6.42470	-12.46792	-12.11013
1.2	-0.45073	-2.84961	-6.77387	-12.74300	-2.27888	-6.60913	-12.67449	-12.15797
1.4	-0.60010	-3.14924	-7.06193	-13.01484	-2.37533	-6.82518	-12.91905	-12.21404
1.5	-0.68058	-3.31466	-7.22430	-13.16728	-2.42819	-6.94479	-13.05562	-12.24510
1.6	-0.76447	-3.49016	-7.39913	-13.33091	-2.48399	-7.07193	-13.20172	-12.27812
1.8	-0.94120	-3.87026	-7.78673	-13.69229	-2.60392	-7.34832	-13.52255	-12.34996
2.0	-1.12773	-4.28713	-8.22572	-14.10020	-2.73411	-7.65315	-13.88150	-12.42929
2.2	-1.32174	-4.73808	-8.71665	-14.55604	-2.87356	-7.98515	-14.27843	-12.51582
2.4	-1.52117	-5.22017	-9.25962	-15.06128	-3.02128	-8.34293	-14.71309	-12.60926
2.5	-1.62238	-5.47190	-9.55048	-15.33292	-3.09794	-8.53104	-14.94443	-12.65847
2.6	-1.72433	-5.73023	-9.85411	-15.61750	-3.17631	-8.72503	-15.18505	-12.70929
2.8	-1.92986	-6.26493	-10.49909	-16.22625	-3.33777	-9.12990	-15.69375	-12.81559
3.0	-2.13673	-6.82088	-11.19294	-16.88903	-3.50480	-9.55600	-16.23847	-12.92785
3.2	-2.34419	-7.39469	-11.93355	-17.60719	-3.67665	-10.00169	-16.81830	-13.04573
3.4	-2.55172	-7.98305	-12.71833	-18.38179	-3.85262	-10.46539	-17.43218	-13.16893
3.5	-2.65540	-8.28171	-13.12637	-18.79050	-3.94196	-10.70349	-17.75150	-13.23242
3.6	-2.75899	-8.58286	-13.54431	-19.21354	-4.03212	-10.94549	-18.07890	-13.29712
3.8	-2.96581	-9.19132	-14.40816	-20.10269	-4.21460	-11.44043	-18.75708	-13.42999
4.0	-3.17207	-9.80595	-15.30630	-21.04897	-4.39960	-11.94872	-19.46527	-13.56726
4.2	-3.37775	-10.42464	-16.23494	-22.05150	-4.58672	-12.46889	-20.20184	-13.70863
4.4	-3.58286	-11.04574	-17.19020	-23.10889	-4.77563	-12.99962	-20.96512	-13.85383
4.5	-3.68521	-11.35675	-17.67658	-23.65755	-4.87066	-13.26854	-21.35624	-13.92778
4.6	-3.78743	-11.66787	-18.16813	-24.21911	-4.96604	-13.53962	-21.75337	-14.00259
4.8	-3.99150	-12.29009	-19.16495	-25.37960	-5.15770	-14.08774	-22.56480	-14.15467
5.0	-4.19514	-12.91171	-20.17691	-26.58737	-5.35043	-14.64294	-23.39762	-14.30983

c	$+d_0$	$-d_2$	$+d_4$	$-d_6$	$+d_8$	$-d_{10}$	$+d_{12}$	$-d_{14}$
0	1.00000	0	0	0	0	0	0	0
0.1	0.99945	.(2)11101	.(6)19029	.(10)13729	0	0	0	0
0.2	.99778	.(2)44290	.(5)30359	.(9)87603	0	0	0	0
0.4	.99122	.017532	.(4)48020	.(7)55398	0	0	0	0
0.5	.98637	.027180	.(3)11623	.(6)20943	.(9)20871	0	0	0
0.6	.98053	.038768	.(3)23849	.(6)61854	.(9)88739	0	0	0
0.8	.96608	.067281	.(3)73390	.(5)33797	.(8)86138	0	0	0
1.0	.94837	.10195	.(2)17316	.(4)12439	.(7)49490	0	0	0
1.2	.92796	.14147	.(2)34449	.(4)35561	.(6)20348	.(9)74042	0	0
1.4	.90547	.18446	.(2)60801	.(4)85205	.(6)66261	.(8)32784	0	0
1.5	.89363	.20680	.(2)78006	.(3)12530	.(5)11176	.(8)63438	0	0
1.6	.88152	.22949	.(2)98150	.(3)17908	.(5)18155	.(7)11718	0	0
1.8	.85672	.27526	.014783	.(3)34007	.(5)43538	.(7)35513	0	0
2.0	.83162	.32059	.021062	.(3)59551	.(5)93876	.(7)94371	0	0
2.2	.80669	.36450	.028673	.(3)97590	.(4)18558	.(6)22528	.(8)18968	0
2.4	.78230	.40625	.037582	.(2)15133	.(4)34127	.(6)49189	.(8)49208	0
2.5	.77040	.42614	.042500	.(2)18509	.(4)45205	.(6)70607	.(8)76575	0
2.6	.75874	.44531	.047710	.(2)22397	.(4)59042	.(6)99610	.(7)11673	0
2.8	.73620	.48135	.058940	.(2)31852	.(4)96952	.(5)18903	.(7)25638	0
3.0	.71480	.51422	.071137	.(2)43773	.(3)15220	.(5)33975	.(7)52774	0
3.2	.69460	.54392	.084149	.(2)58391	.(3)22976	.(5)58145	.(6)10250	.(8)13300
3.4	.67559	.57052	.097824	.(2)75896	.(3)33517	.(5)95380	.(6)18930	.(8)27671
3.5	.66653	.58271	.10486	.(2)85776	.(3)40017	.(4)12042	.(6)25286	.(8)39124
3.6	.65775	.59419	.11201	.(2)96427	.(3)47440	.(4)15071	.(6)33429	.(8)54655
3.8	.64103	.61513	.12657	.012008	.(3)65377	.(4)23036	.(6)56744	.(7)10311
4.0	.62537	.63356	.14138	.014689	.(3)87978	.(4)34182	.(6)92971	.(7)18669
4.2	.61068	.64969	.15631	.017687	.(2)11590	.(4)49391	.(5)14756	.(7)32575
4.4	.59690	.66376	.17127	.020997	.(2)14981	.(4)69681	.(5)22757	.(7)54970
4.5	.59033	.67009	.17873	.022768	.(2)16920	.(4)82085	.(5)27983	.(7)70588
4.6	.58396	.67598	.18616	.024614	.(2)19032	.(4)96202	.(5)34197	.(7)89992
4.8	.57179	.68654	.20092	.028526	.(2)23806	.(3)13023	.(5)50188	.(6)14332
5.0	.56032	.69561	.21548	.032721	.(2)29359	.(3)17318	.(5)72083	.(6)22256

c	$-d_{-2}/\rho$	$+d_{-4}/\rho$	$-d_{-6}/\rho$	$+d_{-8}/\rho$	$-d_{-10}/\rho$	$+d_{-12}/\rho$	$-d_{-14}/\rho$	$+d_{-16}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(2)49906	.(5)16633	.(9)17600	.(14)92306	0	0	0	0
0.2	.019850	.(4)26451	.(7)11194	.(11)23481	0	0	0	0
0.4	.077633	.(3)41305	.(6)69871	.(9)58604	0	0	0	0
0.5	.11929	.(3)99035	.(5)26162	.(8)34277	0	0	0	0
0.6	.16832	.(2)20089	.(5)76368	.(7)14403	0	0	0	0
0.8	.28437	.(2)60071	.(4)40528	.(6)13576	.(9)27224	0	0	0
1.0	.41679	.013674	.(3)14382	.(6)75185	.(8)23540	0	0	0
1.2	.55638	.026095	.(3)39405	.(5)29617	.(7)13340	0	0	0
1.4	.69470	.043946	.(3)89995	.(5)91886	.(7)56261	0	0	0
1.5	.76117	.054986	.(2)12899	.(4)15101	.(6)10607	.(9)49679	0	0
1.6	.82491	.067414	.(2)17952	.(4)23883	.(6)19072	.(8)10158	0	0
1.8	.94219	.096202	.(2)32251	.(4)54148	.(6)54628	.(8)36777	0	0
2.0	1.0437	.12962	.(2)53317	.(3)11014	.(5)13689	.(7)11361	0	0
2.2	1.1285	.16673	.(2)82392	.(3)20515	.(5)30777	.(7)30855	0	0
2.4	1.1970	.20651	.012046	.(3)35535	.(5)63264	.(7)75335	0	0
2.5	1.2255	.22706	.014308	.(3)45686	.(5)88119	.(6)11374	0	0
2.6	1.2504	.24783	.016818	.(3)57932	.(4)12066	.(6)16825	.(8)16803	0
2.8	1.2906	.29006	.022591	.(3)89742	.(4)21600	.(6)34847	.(8)38210	0
3.0	1.3197	.33216	.029369	.(2)13310	.(4)36629	.(6)67636	.(8)89585	0
3.2	1.3395	.37363	.037136	.(2)19018	.(4)59290	.(5)12421	.(7)18677	0
3.4	1.3518	.41401	.045852	.(2)26313	.(4)92169	.(5)21727	.(7)36793	0
3.5	1.3557	.43368	.050552	.(2)30622	.(3)11338	.(5)28273	.(7)50669	0
3.6	1.3583	.45298	.055469	.(2)35404	.(3)13832	.(5)36425	.(7)68906	0
3.8	1.3601	.49030	.065918	.(2)46480	.(3)20122	.(5)58815	.(6)12374	0
4.0	1.3584	.52587	.077132	.(2)59720	.(3)28479	.(5)91858	.(6)21349	0
4.2	1.3539	.55962	.089038	.(2)75282	.(3)39334	.(4)13926	.(6)35569	0
4.4	1.3474	.59150	.10155	.(2)93293	.(3)53146	.(4)20556	.(6)57429	0
4.5	1.3436	.60676	.10802	.010326	.(3)61322	.(4)24750	.(6)72192	0
4.6	1.3394	.62156	.11461	.011388	.(3)70417	.(4)29624	.(6)90128	.(7)20724
4.8	1.3303	.64982	.12812	.013710	.(3)91644	.(4)41766	.(5)13783	.(7)34406
5.0	1.3203	.67633	.14203	.016305	.(2)11737	.(4)57731	.(5)20592	.(7)55602

c	$+d_1$	$-d_3$	$+d_5$	$-d_7$	$+d_9$	$-d_{11}$	$+d_{13}$	$-d_{15}$
0	1.00000	0	0	0	0	0	0	0
0.1	0.99940	.(3)39980	.(7)45330	0	0	0	0	0
0.2	.99760	.(2)15967	.(6)72424	0	0	0	0	0
0.4	.99046	.(2)63479	.(4)11522	.(7)10029	0	0	0	0
0.5	.98514	.(2)98730	.(4)28008	.(7)38279	0	0	0	0
0.6	.97869	.014137	.(4)57774	.(6)11320	0	0	0	0
0.8	.96250	.024776	.(3)18016	.(6)62783	.(8)12901	0	0	0
1.0	.94217	.038007	.(3)43231	.(5)23553	.(8)75651	0	0	0
1.2	.91807	.053514	.(3)87765	.(5)68902	.(7)31881	0	0	0
1.4	.89059	.070931	.(2)15856	.(4)16956	.(6)10684	.(9)44465	0	0
1.5	.87573	.080231	.(2)20604	.(4)25303	.(6)18306	.(9)87475	0	0
1.6	.86021	.089854	.(2)26274	.(4)36726	.(6)30239	.(8)16443	0	0
1.8	.82742	.10985	.(2)40716	.(4)72086	.(6)75153	.(8)51738	0	0
2.0	.79276	.13049	.(2)59797	.(3)13080	.(5)16842	.(7)14318	0	0
2.2	.75676	.15133	.(2)84018	.(3)22251	.(5)34682	.(7)35686	0	0
2.4	.71995	.17195	.011373	.(3)35864	.(5)66545	.(7)81503	.(9)70895	0
2.5	.70141	.18205	.013072	.(3)44734	.(5)90073	.(6)11971	.(8)11299	0
2.6	.68286	.19196	.014913	.(3)55206	.(4)12024	.(6)17285	.(8)17647	0
2.8	.64596	.21103	.019019	.(3)81661	.(4)20627	.(6)34390	.(8)40718	0
3.0	.60970	.22885	.023672	.(2)11665	.(4)33819	.(6)64716	.(8)87953	0
3.2	.57446	.24521	.028834	.(2)16157	.(4)53272	.(5)11595	.(7)17926	0
3.4	.54054	.25993	.034451	.(2)21771	.(4)80981	.(5)19889	.(7)34699	.(9)4524
3.5	.52416	.26665	.037410	.(2)25035	.(4)98637	.(5)25663	.(7)47433	.(9)6552
3.6	.50820	.27292	.040457	.(2)28620	.(3)11924	.(5)32809	.(7)64137	.(9)9371
3.8	.47761	.28414	.046776	.(2)36798	.(3)17059	.(5)52251	.(6)11373	.(8)1850
4.0	.44886	.29359	.053331	.(2)46370	.(3)23780	.(5)80612	.(6)19426	.(8)3499
4.2	.42202	.30135	.060039	.(2)57377	.(3)32376	.(4)12083	.(6)32068	.(8)6364
4.4	.39706	.30750	.066826	.(2)69835	.(3)43144	.(4)17642	.(6)51325	.(7)1117
4.5	.38528	.31001	.070225	.(2)76605	.(3)49436	.(4)21125	.(6)64236	.(7)1461
4.6	.37395	.31216	.073618	.(2)83731	.(3)56382	.(4)25150	.(6)79853	.(7)1897
4.8	.35259	.31547	.080352	.(2)99030	.(3)72379	.(4)35077	.(5)12107	.(7)3127
5.0	.33290	.31756	.086971	.011568	.(3)91415	.(4)47950	.(5)17925	.(7)5016

PROLATE SPHEROIDAL FUNCTIONS m=0, ℓ=1

c	$+d_{-1}/\rho$	$-d_{-3}/\rho$	$+d_{-5}/\rho$	$-d_{-7}/\rho$	$+d_{-9}/\rho$	$-d_{-11}/\rho$	$+d_{-13}/\rho$	$-d_{-15}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(2)16634	.(5)27728	.(9)52818	0	0	0	0	0
0.2	.(2)66154	.(4)44135	.(7)33634	.(11)98932	0	0	0	0
0.4	.025856	.(3)69158	.(5)21094	.(8)25578	0	0	0	0
0.5	.039704	.(2)16622	.(5)79254	.(7)15019	0	0	0	0
0.6	.055979	.(2)33816	.(4)23230	.(7)63409	0	0	0	0
0.8	.094317	.010181	.(3)12451	.(6)60459	0	0	0	0
1.0	.13760	.023359	.(3)44712	.(5)33950	.(7)13946	0	0	0
1.2	.18233	.044914	.(2)12404	.(4)13575	.(7)80344	0	0	0
1.4	.22515	.076149	.(2)28688	.(4)42776	.(6)34480	.(8)17463	0	0
1.5	.24492	.095538	.(2)41366	.(4)70842	.(6)65570	.(8)38131	0	0
1.6	.26319	.11738	.(2)57860	.(3)11290	.(5)11893	.(8)78704	0	0
1.8	.29429	.16780	.010499	.(3)25932	.(5)34594	.(7)28985	0	0
2.0	.31712	.22556	.017464	.(3)53307	.(5)87841	.(7)90895	0	0
2.2	.33123	.28803	.027040	.(3)99959	.(4)19940	.(6)24974	0	0
2.4	.33691	.35211	.039406	.(2)17348	.(4)41201	.(6)61425	0	0
2.5	.33686	.38379	.046645	.(2)22288	.(4)57443	.(6)92933	0	0
2.6	.33505	.41467	.054544	.(2)28195	.(4)78603	.(5)13755	0	0
2.8	.32694	.47286	.072189	.(2)43284	.(3)13996	.(5)28405	0	0
3.0	.31402	.52439	.091898	.(2)63238	.(3)23468	.(5)54667	0	0
3.2	.29779	.56777	.11311	.(2)88495	.(3)37346	.(5)98945		0
3.4	.27959	.60222	.13517	.011924	.(3)56762	.(4)16967	.(6)34851	0
3.5	.27012	.61607	.14633	.013667	.(3)68903	.(4)21818	.(6)47490	0
3.6	.26056	.62771	.15747	.015544	.(3)82857	.(4)27745	.(6)63870	0
3.8	.24157	.64471	.17942	.019684	.(2)11672	.(4)43500	.(5)11148	0
4.0	.22325	.65411	.20056	.024303	.(2)15936	.(4)65717	.(5)18644	0
4.2	.20601	.65699	.22054	.029347	.(2)21164	.(4)96063	.(5)30010	0
4.4	.19007	.65451	.23911	.034758	.(2)27430	.(3)13637	.(5)46690	0
4.5	.18262	.65161	.24782	.037583	.(2)30973	.(3)16089	.(5)57570	0
4.6	.17552	.64779	.25614	.040477	.(2)34795	.(3)18864	.(5)70472	
4.8	.16236	.63782	.27157	.046440	.(2)43301	.(3)25494	.(4)10351	.(6)30730
5.0	.15052	.62550	.28542	.052596	.(2)52981	.(3)33746	.(4)14835	.(6)47707

c	$+d_0$	$+d_2$	$-d_4$	$+d_6$	$-d_8$	$+d_{10}$	$-d_{12}$	$+d_{14}$
0	0	1.0000	0	0	0	0	0	0
0.1	.(3)22221	1.0003	.(5)24496	.(7)20620	0	0	0	0
0.2	.(3)88868	1.0010	.(3)98066	.(6)33020	0	0	0	0
0.4	.(2)35521	1.0042	.(2)39355	.(5)53009	.(8)36906	0	0	0
0.5	.(2)55469	1.0065	.(2)61642	.(4)12974	.(7)14114	0	0	0
0.6	.(2)79812	1.0093	.(2)89030	.(4)26985	.(7)42277	0	0	0
0.8	.014157	1.0163	.015947	.(4)85952	.(6)23942	0	0	0
1.0	.022044	1.0251	.025156	.(3)21193	.(6)92262	.(8)24727	0	0
1.2	.031586	1.0354	.036640	.(3)44476	.(5)26936	.(7)10766	0	0
1.4	.042694	1.0470	.050529	.(3)83549	.(5)71341	.(7)37492	0	0
1.5	.048797	1.0531	.058418	.(2)11094	.(4)10877	.(7)65634	0	0
1.6	.055240	1.0594	.066960	.(2)14476	.(4)16155	.(6)11093	.(9)51780	0
1.8	.069055	1.0721	.086075	.(2)23584	.(4)33334	.(6)28982	.(8)17128	0
2.0	.083921	1.0845	.10800	.(2)36596	.(4)63916	.(6)68648	.(8)50105	0
2.2	.099577	1.0961	.13283	.(2)54577	.(3)11547	.(5)15017	.(7)13268	0
2.4	.11573	1.1060	.16061	.(2)78736	.(3)19851	.(5)30748	.(7)32352	0
2.5	.12388	1.1101	.17561	.(2)93544	.(3)25609	.(5)43063	.(7)49178	0
2.6	.13205	1.1135	.19134	.011040	.(3)32717	.(5)59434	.(7)73561	.(9)6588
2.8	.14820	1.1180	.22492	.015101	.(3)51989	.(4)10983	.(6)15751	.(8)1637
3.0	.16387	1.1188	.26116	.020202	.(3)79994	.(4)19423	.(6)32000	.(8)3819
3.2	.17875	1.1155	.29979	.026491	.(2)11959	.(4)33079	.(6)62061	.(8)8433
3.4	.19256	1.1075	.34044	.034106	.(2)17419	.(4)54464	.(5)11546	.(7)1772
3.5	.19900	1.1018	.36139	.038450	.(2)20833	.(4)69073	.(5)15524	.(7)2526
3.6	.20510	1.0949	.38267	.043171	.(2)24773	.(4)86956	.(5)20685	.(7)3562
3.8	.21620	1.0774	.42595	.053788	.(2)34470	.(3)13499	.(5)35815	.(7)6875
4.0	.22575	1.0554	.46970	.066025	.(2)46988	.(3)20416	.(5)60066	.(6)1278
4.2	.23372	1.0290	.51333	.079920	.(2)62843	.(3)30143	.(5)97852	.(6)2297
4.4	.24012	.99875	.55623	.095469	.(2)82557	.(3)43511	.(4)15514	.(6)3999
4.5	.24274	.98235	.57723	.10385	.(2)94025	.(3)51861	.(4)19348	.(6)5217
4.6	.24500	.96520	.59784	.11263	.010665	.(3)61498	.(4)23982	.(6)6759
4.8	.24848	.92897	.63760	.13131	.013561	.(3)85219	.(4)36205	.(5)1111
5.0	.25069	.89072	.67508	.15139	.016987	.(2)11591	.(4)53456	.(5)1781

c	$+d_{-2}/\rho$	$-d_{-4}/\rho$	$+d_{-6}/\rho$	$-d_{-8}/\rho$	$+d_{-10}/\rho$	$-d_{-12}/\rho$	$+d_{-14}/\rho$	$-d_{-16}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(6)55563	.(9)34135	0	0	0	0	0	0
0.2	.(5)88935	.(7)23718	.(10)39531	0	0	0	0	0
0.4	.(3)14250	.(5)15206	.(8)31242	.(11)23947	0	0	0	0
0.5	.(3)34822	.(5)58075	.(7)19209	.(10)28284	0	0	0	0
0.6	.(3)72276	.(4)17364	.(7)82715	.(9)16943	0	0	0	0
0.8	.(2)22880	.(4)97821	.(6)82871	.(8)31368	0	0	0	0
1.0	.(2)55879	.(3)37409	.(5)49546	.(7)29122	.(10)97927	0	0	0
1.2	.011562	.(2)11180	.(4)21340	.(6)18070	.(9)87520	.(11)27587	0	0
1.4	.021291	.(2)28151	.(4)73223	.(6)84439	.(8)55684	.(10)23896	0	0
1.5	.027916	.(2)42496	.(3)12698	.(5)16816	.(7)12733	.(10)62734	0	0
1.6	.035896	.(2)62388	.(3)21229	.(5)31999	.(7)27574	.(9)15460	0	0
1.8	.056415	.012517	.(3)54018	.(4)10315	.(6)11256	.(9)79902	0	0
2.0	.083593	.023158	.(2)12371	.(4)29201	.(6)39369	.(8)34518	0	0
2.2	.11770	.040021	.(2)25952	.(4)74234	.(5)12121	.(7)12867	0	0
2.4	.15833	.065209	.(2)50519	.(3)17228	.(5)33510	.(7)41214	0	0
2.5	.18072	.081588	.(2)68745	.(3)27185	.(5)57409	.(7)78778	.(9)76084	0
2.6	.20425	.10085	.(2)92120	.(3)36944	.(5)84437	.(6)12537	.(8)13100	0
2.8	.25344	.14885	.015849	.(3)73888	.(4)19609	.(6)36050	.(8)43714	0
3.0	.30318	.21050	.025885	.(2)13888	.(4)42378	.(6)83926	.(7)11690	.(9)12094
3.2	.35032	.28646	.040321	.(2)24683	.(4)85830	.(5)19360	.(7)30715	.(9)36162
3.4	.39157	.37592	.060143	.(2)41685	.(3)16391	.(5)41782	.(7)74866	.(9)99590
3.5	.40906	.42523	.072349	.(2)53218	.(3)22193	.(5)59980	.(6)10976	.(8)15476
3.6	.42404	.47718	.086208	.(2)67191	.(3)29670	.(5)84883	.(6)17065	.(8)25464
3.8	.44548	.58738	.11913	.010378	.(3)51148	.(4)16322	.(6)36588	.(8)60862
4.0	.45451	.70272	.15914	.015408	.(3)84284	.(4)29834	.(6)74157	.(7)13675
4.2	.45087	.81903	.20610	.022068	.(2)13331	.(4)52077	.(5)14281	.(7)29048
4.4	.43522	.93199	.25941	.030573	.(2)20300	.(4)87115	.(5)26236	.(7)58592
4.5	.42332	.98593	.28814	.035569	.(2)24720	.(3)11101	.(5)34980	.(7)81725
4.6	.40904	1.0376	.31807	.041082	.(2)29854	.(3)14014	.(5)46155	.(6)11270
4.8	.37440	1.1328	.38088	.053696	.(2)42538	.(3)21757	.(5)78059	.(6)20758
5.0	.33360	1.2151	.44637	.068424	.(2)58871	.(3)32690	.(4)12730	.(6)36737

c	$+d_1$	$+d_3$	$-d_5$	$+d_7$	$-d_9$	$+d_{11}$	$-d_{13}$	$+d_{15}$
0	0	1.00000	0	0	0	0	0	0
0.1	.(3)14285	0.99988	.(3)17635	0	0	0	0	0
0.2	.(3)68567	.99958	.(3)70518	.(6)18829	0	0	0	0
0.4	.(2)27421	.99830	.(2)28173	.(5)30090	0	0	0	0
0.5	.(2)42838	.99735	.(2)43980	.(5)73397	0	0	0	0
0.6	.(2)61673	.99618	.(2)63260	.(4)15261	0	0	0	0
0.8	.010958	.99322	.011215	.(4)47919	.(7)70462	0	0	0
1.0	.017108	.98941	.017461	.(3)11659	.(6)42212	0	0	0
1.2	.024607	.98476	.025037	.(3)24078	.(5)12555	0	0	0
1.4	.033442	.97925	.033911	.(3)44401	.(5)31516	0	0	0
1.5	.038353	.97618	.038823	.(3)58364	.(5)47562	0	0	0
1.6	.043590	.97289	.044047	.(3)75355	.(5)69902	0	0	0
1.8	.055020	.96565	.055406	.(2)12003	.(4)14091	.(6)10495	0	0
2.0	.067686	.95749	.067949	.(2)18186	.(4)26367	.(6)24251	0	0
2.2	.081525	.94838	.081631	.(2)26460	.(4)46443	.(6)51702	0	0
2.4	.096451	.93823	.096408	.(2)37233	.(4)77823	.(5)10315	0	0
2.5	.10429	.93275	.10419	.(2)43692	.(4)99130	.(5)14260	0	0
2.6	.11235	.92697	.11223	.(2)50943	.(3)12508	.(5)19466	0	0
2.8	.12908	.91450	.12904	.(2)68054	.(3)19395	.(5)35028	0	0
3.0	.14644	.90068	.14678	.(2)89060	.(3)29172	.(5)60525	.(7)87140	0
3.2	.16424	.88536	.16537	.011447	.(3)42719	.(4)10093	.(6)16544	0
3.4	.18219	.86841	.18472	.014479	.(3)61104	.(4)16315	.(6)30211	0
3.5	.19113	.85927	.19464	.016197	.(3)74390	.(4)21060	.(6)41342	0
3.6	.20002	.84966	.20472	.018056	.(3)85588	.(4)25651	.(6)53295	0
3.8	.21741	.82899	.22523	.022226	.(2)11764	.(4)39338	.(6)91152	0
4.0	.23404	.80626	.24610	.027035	.(2)15895	.(4)58983	.(5)15160	0
4.2	.24959	.78142	.26713	.032524	.(2)21140	.(4)86634	.(5)24578	.(7)51274
4.4	.26377	.75444	.28812	.038724	.(2)27707	.(3)12485	.(5)38921	.(7)89191
4.5	.27026	.74016	.29853	.042089	.(2)31549	.(3)14884	.(5)48567	.(6)11647
4.6	.27630	.72537	.30883	.045657	.(2)35820	.(3)17676	.(5)60312	.(6)15120
4.8	.28699	.69431	.32903	.053327	.(2)45713	.(3)24614	.(5)91579	.(6)25023
5.0	.29568	.66145	.34845	.061728	.(2)57625	.(3)33743	.(4)13643	.(6)40488

c	$+d_{-1}/\rho$	$+d_{-3}/\rho$	$-d_{-5}/\rho$	$+d_{-7}/\rho$	$-d_{-9}/\rho$	$+d_{-11}/\rho$	$-d_{-13}/\rho$	$+d_{-15}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(8)39675	.(11)44084	0	0	0	0	0	0
0.2	.(6)76140	.(8)33842	.(11)58017	0	0	0	0	0
0.4	.(4)12159	.(6)21622	.(8)14828	0	0	0	0	0
0.5	.(4)29641	.(6)82370	.(8)88270	0	0	0	0	0
0.6	.(4)61353	.(5)24554	.(7)37894	.(9)17570	0	0	0	0
0.8	.(3)19306	.(4)13736	.(6)37700	.(8)24377	0	0	0	0
1.0	.(3)46873	.(4)52091	.(5)22353	.(7)22588	0	0	0	0
1.2	.(3)96558	.(3)15435	.(5)95472	.(6)13897	0	0	0	0
1.4	.(2)17756	.(3)38543	.(4)32499	.(6)64412	.(8)60492	0	0	0
1.5	.(2)23305	.(3)57970	.(4)56169	.(5)12783	.(7)13776	0	0	0
1.6	.(2)30043	.(3)84837	.(4)93642	.(5)24255	.(7)29745	0	0	0
1.8	.(2)47694	.(2)16939	.(3)23739	.(5)77881	.(6)12092	0	0	0
2.0	.(2)71996	.(2)31282	.(3)54362	.(4)22041	.(6)42271	0	0	0
2.2	.010432	.(2)54154	.(2)11454	.(4)56271	.(5)13067	0	0	0
2.4	.014611	.(2)88738	.(2)22510	.(3)13184	.(5)36582	0	0	0
2.5	.017133	.011175	.(2)30902	.(3)19659	.(5)56027	0	0	0
2.6	.019878	.013860	.(2)41672	.(3)28709	.(5)93287	.(6)17976	0	0
2.8	.026372	.020737	.(2)73220	.(3)58664	.(4)22137	.(6)49509	0	0
3.0	.034208	.029831	.012281	.(2)11334	.(4)49171	.(5)12636	0	0
3.2	.043475	.041366	.019752	.(2)20824	.(3)10298	.(5)30143	0	0
3.4	.054197	.055384	.030559	.(2)36547	.(3)20449	.(5)67658	0	0
3.5	.060094	.063271	.037494	.(2)47646	.(3)28285	.(5)99242	.(6)23178	0
3.6	.066334	.071681	.045605	.(2)61491	.(3)38672	.(4)14366	.(6)35513	0
3.8	.079743	.089729	.065784	.(2)99472	.(3)69905	.(4)28985	.(6)79951	0
4.0	.094175	.10866	.091870	.015507	.(2)12115	.(4)55759	.(5)17064	0
4.2	.10926	.12728	.12439	.023341	.(2)20179	.(3)10262	.(5)34672	0
4.4	.12454	.14416	.16349	.033983	.(2)32374	.(3)18111	.(5)67260	0
4.5	.13208	.15146	.18544	.040517	.(2)40460	.(3)23704	.(5)92122	.
4.6	.13946	.15774	.20885	.047931	.(2)50127	.(3)30726	.(4)12492	.(6)3629
4.8	.15343	.16655	.25956	.065582	.(2)75032	.(3)50212	.(4)22268	.(6)7052
5.0	.16592	.16936	.31423	.087187	.010878	.(3)79210	.(4)38182	.(5)1313

c	$+d_0$	$-d_2$	$+d_4$	$-d_6$	$+d_8$	$-d_{10}$	$+d_{12}$	$-d_{14}$
0	1.0000	0	0	0	0	0	0	0
0.1	0.99980	.(3)13327	.(8)90651	0	0	0	0	0
0.2	.99920	.(3)53234	.(6)14479	0	0	0	0	0
0.4	.99682	.(2)21175	.(5)23010	.(8)14290	0	0	0	0
0.5	.99505	.(2)32948	.(5)55890	.(8)54210	0	0	0	0
0.6	.99290	.(2)47204	.(4)11518	.(7)16077	0	0	0	0
0.8	.98751	.(2)82845	.(4)35831	.(7)88781	0	0	0	0
1.0	.98074	.012734	.(4)85735	.(6)33126	.(9)82303	0	0	0
1.2	.97272	.017977	.(3)17350	.(6)96294	.(8)34434	0	0	0
1.4	.96354	.023910	.(3)31238	.(5)23530	.(7)11432	0	0	0
1.5	.95858	.027100	.(3)40521	.(5)34981	.(7)19490	0	0	0
1.6	.95339	.030421	.(3)51584	.(5)50579	.(7)32028	0	0	0
1.8	.94239	.037394	.(3)79679	.(5)98495	.(7)78746	0	0	0
2.0	.93070	.044715	.(2)11668	.(4)17730	.(6)17452	.(8)11997	0	0
2.2	.91846	.052271	.(2)16358	.(4)29929	.(6)35537	.(8)29498	0	0
2.4	.90581	.059959	.(2)22113	.(4)47889	.(6)67444	.(8)66469	0	0
2.5	.89937	.063821	.(2)25407	.(4)59532	.(6)90807	.(8)96985	0	0
2.6	.89289	.067682	.(2)28984	.(4)73235	.(5)12059	.(7)13913	0	0
2.8	.87979	.075357	.(2)36998	.(3)10773	.(5)20490	.(7)27339	0	0
3.0	.86664	.082910	.(2)46155	.(3)15321	.(5)33306	.(7)50860	.(9)57435	0
3.2	.85352	.090280	.(2)56436	.(3)21157	.(5)52082	.(7)90193	.(8)11561	0
3.4	.84052	.097420	.(2)67797	.(3)28466	.(5)78707	.(6)15333	.(8)22130	0
3.5	.83407	.10089	.(2)73866	.(3)32729	.(5)95640	.(6)19707	.(8)30100	0
3.6	.82768	.10429	.(2)80182	.(3)37427	.(4)11539	.(6)25106	.(8)40512	0
3.8	.81507	.11086	.(2)93518	.(3)48205	.(4)16464	.(6)39753	.(8)71262	0
4.0	.80272	.11712	.010772	.(3)60952	.(4)22926	.(6)61077	.(7)12094	0
4.2	.79067	.12306	.012271	.(3)75799	.(4)31232	.(6)91322	.(7)19870	0
4.4	.77893	.12866	.013839	.(3)92858	.(4)41709	.(5)13322	.(7)31701	.(9)5826
4.5	.77318	.13134	.014646	.(2)10225	.(4)47870	.(5)15953	.(7)39635	.(9)7608
4.6	.76751	.13393	.015466	.(2)11222	.(4)54704	.(5)19002	.(7)49240	.(9)9862
4.8	.75643	.13888	.017144	.(2)13394	.(4)70575	.(5)26555	.(7)74637	.(8)1623
5.0	.74568	.14351	.018863	.(2)15808	.(4)89688	.(5)36452	.(6)11062	.(8)2601

c	d_{-2}	d_{-4}/ρ	d_{-6}/ρ	d_{-8}/ρ	d_{-10}/ρ	d_{-12}/ρ	d_{-14}/ρ	d_{-16}/ρ
0	0	0	0	0	0	0	0	0
0.1	.(2)33461	.(5)27868	-.(9)26537	0	0	0	0	0
0.2	.013539	.(4)45027	-.(7)17142	.(11)34620	0	0	0	0
0.4	.056803	.(3)74951	-.(5)11393	.(9)91957	-.(12)45241	0	0	0
0.5	.092149	.(2)18927	-.(5)44889	.(8)56575	-.(11)43473	0	0	0
0.6	.13924	.(2)40928	-.(4)13954	.(7)25304	-.(10)27986	0	0	0
0.8	.28351	.014585	-.(4)88015	.(6)28316	-.(9)55610	0	0	0
1.0	.54688	.043095	-.(3)40407	.(5)20258	-.(8)62068	.(10)12775	0	0
1.2	1.1132	.12334	-.(2)16538	.(4)11901	-.(7)52407	.(9)15513	0	0
1.4	3.0315	.44473	-.(2)80508	.(4)78549	-.(6)46974	.(8)18899	-.(11)54568	0
1.5	8.0693	1.3385	-.027689	.(3)30945	-.(5)21217	.(8)97908	-.(10)32433	0
	∞	∞	∞	∞	∞	∞	∞	∞
1.6	-21.406	-3.9761	.093130	-.(2)11814	.(5)92034	-.(7)48278	.(9)18184	0
1.8	-3.2002	-.72694	.021320	-.(3)34055	.(5)33475	-.(7)22180	.(9)10559	0
2.0	-1.9606	-.52974	.018954	-.(3)37616	.(5)44143	-.(7)36682	.(9)21524	0
2.2	-1.5071	-.47350	.020232	-.(3)47688	.(5)69521	-.(7)68485	.(9)48538	0
2.4	-1.2700	-.45531	.022822	-.(3)63565	.(4)10982	-.(6)12839	.(8)10808	0
2.5	-1.1889	-.45252	.024425	-.(3)73537	.(4)13754	-.(6)17492	.(8)15960	-.(10)1098
2.6	-1.1228	-.45209	.026185	-.(3)84932	.(4)17141	-.(6)23447	.(8)23114	-.(10)1718
2.8	-1.0213	-.45565	.030103	-.(2)11229	.(4)26151	-.(6)41348	.(8)47160	-.(10)4058
3.0	-.94625	-.46238	.034451	-.(2)14619	.(4)38871	-.(6)70297	.(8)91805	-.(10)9052
3.2	-.88776	-.47042	.039137	-.(2)18714	.(4)56285	-.(5)11536	.(7)17093	-.(9)19135
3.4	-.84044	-.47880	.044090	-.(2)23555	.(4)79477	-.(5)18313	.(7)30541	-.(9)38510
3.5	-.81988	-.48292	.046644	-.(2)26265	.(4)93602	-.(5)22804	.(7)40238	-.(9)53703
3.6	-.80099	-.48694	.049242	-.(2)29173	.(3)10962	-.(5)28190	.(7)52539	-.(9)75504
3.8	-.76734	-.49454	.054536	-.(2)35587	.(3)14794	-.(5)42188	.(7)87307	-.(8)13683
4.0	-.73811	-.50144	.059921	-.(2)42805	.(3)19570	-.(5)61525	.(6)14057	-.(8)25289
4.2	-.71233	-.50756	.065350	-.(2)50822	.(3)25416	-.(5)87625	.(6)21988	-.(8)41862
4.4	-.68933	-.51289	.070783	-.(2)59624	.(3)32457	-.(4)12212	.(6)33496	-.(8)69777
4.5	-.67870	-.51533	.073499	-.(2)64322	.(3)36469	-.(4)14311	.(6)40971	-.(8)89133
4.6	-.66858	-.51744	.076184	-.(2)69189	.(3)40813	-.(4)16686	.(6)49810	-.(7)11305
4.8	-.64973	-.52127	.081527	-.(2)79489	.(3)50598	-.(4)22386	.(6)72439	-.(7)17841
5.0	-.63248	-.52442	.086787	-.(2)90488	.(3)61920	-.(4)29537	.(5)10319	-.(7)21590

c	$+d_1$	$-d_3$	$+d_5$	$-d_7$	$+d_9$	$-d_{11}$	$+d_{13}$	$-d_{15}$
0	1.0000	0	0	0	0	0	0	0
0.1	0.99969	.(3)12241	.(8)69690	0	0	0	0	0
0.2	.99878	.(3)48912	.(6)10978	0	0	0	0	0
0.4	.99512	.(2)19485	.(5)17490	.(9)91307	0	0	0	0
0.5	.99239	.(2)30351	.(5)42561	.(8)34715	0	0	0	0
0.6	.98908	.(2)43541	.(5)87906	.(7)10324	0	0	0	0
0.8	.98071	.(2)76669	.(4)27504	.(7)57406	0	0	0	0
1.0	.97013	.011833	.(4)66284	.(6)21608	.(9)46283	0	0	0
1.2	.95744	.016787	.(3)13529	.(6)63479	.(8)19573	0	0	0
1.4	.94279	.022450	.(3)24601	.(5)15702	.(8)65870	0	0	0
1.5	.93478	.025521	.(3)32085	.(5)23500	.(7)11314	0	0	0
1.6	.92634	.028735	.(3)41077	.(5)34218	.(7)18740	0	0	0
1.8	.90827	.035548	.(3)64219	.(5)67646	.(7)46862	0	0	0
2.0	.88878	.042790	.(3)95266	.(4)12376	.(6)10578	.(9)64097	0	0
2.2	.86804	.050357	.(2)13538	.(4)21256	.(6)21965	.(8)16096	0	0
2.4	.84628	.058148	.(2)18561	.(4)34634	.(6)42552	.(8)37087	0	0
2.5	.83507	.062096	.(2)21480	.(4)43456	.(6)57904	.(8)54740	0	0
2.6	.82369	.066062	.(2)24682	.(4)53966	.(6)77734	.(8)79452	0	0
2.8	.80047	.074002	.(2)31969	.(4)80919	.(5)13502	.(7)15991	0	0
3.0	.77681	.081877	.(2)40466	.(3)11734	.(5)22445	.(7)30488	0	0
3.2	.75291	.089604	.(2)50193	.(3)16522	.(5)35904	.(7)55428	.(9)63964	0
3.4	.72894	.097108	.(2)61146	.(3)22664	.(5)55503	.(7)96613	.(8)12575	0
3.5	.71697	.10076	.(2)67075	.(3)26308	.(5)68211	.(6)12574	.(8)17334	0
3.6	.70505	.10433	.(2)73298	.(3)30371	.(5)83226	.(6)16220	.(8)23644	0
3.8	.68138	.11120	.(2)86596	.(3)39852	.(4)12142	.(6)26325	.(8)42709	0
4.0	.65807	.11769	.010097	.(3)51308	.(4)17281	.(6)41445	.(8)74409	0
4.2	.63521	.12376	.011633	.(3)64924	.(4)24047	.(6)63466	.(7)12545	0
4.4	.61290	.12939	.013258	.(3)80869	.(4)32780	.(6) 94764	.(7)20527	.(9)3455
4.5	.60198	.13204	.014100	.(3)89760	.(4)38001	.(5)11478	.(7)25985	.(9)4572
4.6	.59121	.13456	.014960	.(3)99284	.(4)43854	.(5)13826	.(7)32679	.(9)6004
4.8	.57020	.13927	.016727	.(2)12028	.(4)57662	.(5)19747	.(7)50731	.(8)1013
5.0	.54991	.14352	.018547	.(2)14397	.(4)74621	.(5)27658	.(7)76951	.(8)1665

c	d_{-1}	d_{-3}/ρ	d_{-5}/ρ	d_{-7}/ρ	d_{-9}/ρ	d_{-11}/ρ	d_{-13}/ρ	d_{-15}/ρ
0	0	0	0	0	0	0	0	0
0.1	.(3)66708	-.(5)11114	.(9)24696	0	0	0	0	0
0.2	.(2)26736	-.(4)26675	.(7)23705	-.(11)75247	0	0	0	0
0.4	.010779	-.(3)42725	.(5)15175	-.(8)19262	0	0	0	0
0.5	.016944	-.(2)10440	.(5)57907	-.(7)11482	0	0	0	0
0.6	.024583	-.(2)21676	.(4)17301	-.(7)49386	0	0	0	0
0.8	.044572	-.(2)68774	.(4)97407	-.(6)49398	0	0	0	0
1.0	.071533	-.016902	.(3)37314	-.(5)29539	.(7)12377	0	0	0
1.2	.10669	-.035426	.(2)11228	-.(4)12787	.(7)77104	0	0	0
1.4	.15193	-.066736	.(2)28684	-.(4)44388	.(6)36402	0	0	0
1.5	.17920	-.088951	.(2)43797	-.(4)77738	.(6)73153	0	0	0
1.6	.21024	-.11677	.(2)65264	-.(3)13168	.(5)14092	-.(8)94097	0	0
1.8	.28652	-.19427	.013672	-.(3)34841	.(5)47136	-.(7)39807	0	0
2.0	.38925	-.31318	.027047	-.(3)84889	.(4)14159	-.(6)14750	0	0
2.2	.53431	-.49833	.051720	-.(2)19586	.(4)39467	-.(6)49701	0	0
2.4	.75487	-.80031	.098080	-.(2)44059	.(3)10548	-.(5)15791	.(7)16217	0
2.5	.91525	-1.0280	.13612	-.(2)66226	.(3)17187	-.(5)27901	.(7)31079	0
2.6	1.1335	-1.3435	.19154	-.010061	.(3)28211	-.(5)49502	.(7)59613	0
2.8	1.9461	-2.5421	.41621	-.025249	.(3)81921	-.(4)16647	.(6)23227	0
3.0	5.0298	-7.1518	1.3296	-.092161	.(2)34238	-.(4)79736	.(5)18569	0
	∞	∞	∞	∞	∞	∞	∞	∞
3.2	-12.941	19.816	-4.1410	.32486	-.013691	.(3)36211	-.(5)65831	.(7)87383
3.4	-3.0796	5.0310	-1.1712	.10312	-.(2)48902	.(3)14571	-.(5)29862	.(7)44699
3.5	-2.2717	3.8214	-.93607	.087060	-.(2)43672	.(3)13774	-.(5)29889	.(7)47382
3.6	-1.8143	3.1365	-.80684	.079125	-.(2)41914	.(3)13969	-.(5)32043	.(7)53707
3.8	-1.3121	2.3834	-.67251	.072966	-.(2)42893	.(3)15887	-.(5)40532	.(7)75590
4.0	-1.0396	1.9719	-.60620	.072313	-.(2)46896	.(3)19192	-.(5)54149	.(6)11177
4.2	-.86674	1.7075	-.56836	.074121	-.(2)52739	.(3)23721	-.(5)73627	.(6)16728
4.4	-.74623	1.5197	-.54462	.077243	-.(2)60004	.(3)29521	-.(4)10033	.(6)24974
4.5	-.69843	1.4439	-.53585	.079112	-.(2)64105	.(3)32928	-.(4)11690	.(6)30409
4.6	-.65666	1.3768	-.52842	.081108	-.(2)68476	.(3)36685	-.(4)13592	.(6)36910
4.8	-.58699	1.2626	-.51651	.085433	-.(2)78057	.(3)45353	-.(4)18246	.(6)53840
5.0	-.53095	1.1681	-.50705	.089995	-.(2)88639	.(3)55646	-.(4)24218	.(6)77371

c	$+d_0$	$+d_2$	$-d_4$	$+d_6$	$-d_8$	$+d_{10}$	$-d_{12}$	$+d_{14}$
0	0	1.00000	0	0	0	0	0	0
0.1	.(3)34279	1.00010	.(3)10583	0	0	0	0	0
0.2	.(2)13705	1.00038	.(3)42342	.(7)80753	0	0	0	0
0.4	.(2)54707	1.00153	.(2)16954	.(5)12933	0	0	0	0
0.5	.(2)85348	1.00237	.(2)26510	.(5)31596	.(8)22237	0	0	0
0.6	.012267	1.00339	.(2)38210	.(5)65572	.(8)66454	0	0	0
0.8	.021700	1.00593	.(2)68081	.(4)20767	.(7)37412	0	0	0
1.0	.033693	1.00905	.010667	.(4)50833	.(6)14307	0	0	0
1.2	.048137	1.01267	.015410	.(3)10572	.(6)42843	0	0	0
1.4	.064899	1.01666	.021051	.(3)19652	.(5)10838	.(8)39911	0	0
1.5	.074102	1.01876	.024211	.(3)25944	.(5)16423	.(8)60478	0	0
1.6	.083823	1.02088	.027600	.(3)33645	.(5)24231	.(7)11653	0	0
1.8	.10472	1.02518	.035068	.(3)54091	.(5)49293	.(7)30000	0	0
2.0	.12740	1.02938	.043462	.(3)82742	.(5)93072	.(7)69920	0	0
2.2	.15160	1.03329	.052783	.(2)12156	.(4)16541	.(6)15034	.(9)98187	0
2.4	.17709	1.03672	.063023	.(2)17268	.(4)27959	.(6)30237	.(8)17743	0
2.5	.19023	1.03819	.068484	.(2)19609	.(4)35763	.(6)41962	.(8)35383	0
2.6	.20360	1.03947	.074169	.(2)23844	.(4)45299	.(6)57484	.(8)52423	0
2.8	.23084	1.04135	.086193	.(2)32129	.(4)70771	.(5)10414	.(7)11013	0
3.0	.25854	1.04218	.099059	.(2)42376	.(3)10713	.(5)18092	.(7)21960	0
3.2	.28641	1.04178	.11272	.(2)54845	.(3)15771	.(5)30297	.(7)41832	0
3.4	.31417	1.04002	.12710	.(2)69793	.(3)22649	.(5)49106	.(7)76528	0
3.5	.32794	1.03858	.13454	.(2)78275	.(3)26912	.(5)61823	.(6)10208	.(8)12729
3.6	.34157	1.03675	.14214	.(2)87469	.(3)31810	.(5)77296	.(6)13502	.(8)17809
3.8	.36836	1.03190	.15774	.010810	.(3)43784	.(4)11850	.(6)23056	.(8)33876
4.0	.39433	1.02543	.17380	.013191	.(3)59168	.(4)17736	.(6)38224	.(8)62210
4.2	.41925	1.01720	.19021	.015906	.(3)78608	.(4)25967	.(6)61674	.(7)11062
4.4	.44301	1.00734	.20687	.018970	.(2)10282	.(4)37255	.(6)97071	.(7)19102
4.5	.45440	1.00178	.21525	.020636	.(2)11694	.(4)44306	.(5)12072	.(7)24841
4.6	.46546	0.99583	.22364	.022393	.(2)13254	.(4)52454	.(5)14931	.(7)32013
4.8	.48651	.98274	.24041	.026181	.(2)16856	.(4)72580	.(5)22481	.(7)52592
5.0	.50609	.96817	.25706	.030334	.(2)21167	.(4)98805	.(5)33184	.(7)84184

c	d_{-2}	d_{-4}/ρ	d_{-6}/ρ	d_{-8}/ρ	d_{-10}/ρ	d_{-12}/ρ	d_{-14}/ρ
0	0	0	0	0	0	0	0
0.1	.(7)19052	0	0	0	0	0	0
0.2	.(5)30510	-.(8)67783	0	0	0	0	0
0.4	.(4)48977	-.(6)43489	.(8)14905	0	0	0	0
0.5	.(3)11987	-.(5)16621	.(8)89005	-.(10)14981	0	0	0
0.6	.(3)24933	-.(5)49744	.(7)38352	-.(10)92948	0	0	0
0.8	.(3)79411	-.(4)28103	.(6)38505	-.(8)16587	0	0	0
1.0	.(2)19587	-.(3)10823	.(5)23161	-.(7)15585	0	0	0
1.2	.(2)41133	-.(3)32519	.(4)10018	-.(7)97046	0	0	0
1.4	.(2)77381	-.(3)82808	.(4)34714	-.(6)45758	0	0	0
1.5	.010287	-.(2)12595	.(4)60608	-.(6)91696	0	0	0
1.6	.013443	-.(2)18656	.(3)10215	-.(5)17581	.(7)16140	0	0
1.8	.021999	-.(2)38288	.(3)26538	-.(5)57791	.(7)67133	0	0
2.0	.034379	-.(2)73040	.(3)62535	-.(4)16808	.(6)24099	0	0
2.2	.051827	-.013140	.(2)13626	-.(4)44302	.(6)76842	0	0
2.4	.075954	-.022538	.(2)27357	-.(3)10776	.(5)22239	-.(7)29097	0
2.5	.091167	-.029076	.(2)39033	-.(3)16382	.(5)36679	-.(7)52068	0
2.6	.10891	-.037185	.(2)54047	-.(3)24532	.(5)59401	-.(7)91105	0
2.8	.15363	-.059441	.010046	-.(3)52874	.(4)14844	-.(6)26424	0
3.0	.21423	-.092640	.018032	-.(2)10893	.(4)35097	-.(6)71705	0
3.2	.29722	-.14184	.031536	-.(2)21672	.(4)79423	-.(5)18457	0
3.4	.41232	-.21461	.054114	-.(2)41974	.(3)17350	-.(5)45504	.(7)83009
3.5	.48680	-.26351	.070591	-.(2)58018	.(3)25422	-.(5)70643	.(6)13654
3.6	.57653	-.32372	.091998	-.(2)79986	.(3)37071	-.(4)10896	.(6)22276
3.8	.82140	-.49244	.15687	-.015192	.(3)78414	-.(4)25670	.(6)58450
4.0	1.2138	-.76918	.27383	-.029375	.(2)16797	-.(4)60878	.(5)15352
4.2	1.9246	-1.2784	.50454	-.059647	.(2)37563	-.(3)15007	.(5)41701
4.4	3.5622	-2.4627	1.0754	-.13945	.(2)96306	-.(3)42199	.(4)12861
4.5	5.5454	-3.8884	1.7836	-.24184	.017462	-.(3)80003	.(4)25495
4.6	11.055	-7.8684	3.7883	-.53652	.040458	-.(2)19362	.(4)64449
......	∞	∞	∞	∞	∞	∞	∞
4.8	-15.726	11.361	-6.1071	.94088	-.77171	.040176	-.(2)14549
5.0	-5.2638	3.8859	-2.2538	.37632	-.033448	.(2)18876	-.(4)74096

c	$-A_{2,0}$	$+d_0$	$-d_2$	$+d_4$	$-d_6$	$+d_8$	$-d_{10}$	$+d_{12}$
0	6.00000	1.00000	0	0	0	0	0	0
0.1	6.00143	.99990	.(4)40805	.(8)13738	0	0	0	0
0.2	6.00571	.99959	.(3)16308	.(7)21956	0	0	0	0
0.4	6.02281	.99837	.(3)65007	.(6)34972	0	0	0	0
0.5	6.03559	.99746	.(2)10131	.(6)85095	.(9)49527	0	0	0
0.6	6.05118	.99636	.(2)14543	.(5)17573	.(8)14720	0	0	0
0.8	6.09064	.99356	.(2)25649	.(5)54966	.(8)81738	0	0	0
1.0	6.14095	.99002	.(2)39671	.(4)13242	.(7)30713	0	0	0
1.2	6.20179	.98577	.(2)56427	.(4)27020	.(7)90043	0	0	0
1.4	6.27280	.98086	.(2)75704	.(4)49120	.(6)22222	.(9)72142	0	0
1.5	6.31199	.97816	.(2)86215	.(4)64057	.(6)33002	.(8)12288	0	0
1.6	6.35356	.97532	.(2)97266	.(4)82005	.(6)48309	.(8)20444	0	0
1.8	6.44362	.96922	.012086	.(3)12821	.(6)95235	.(8)50908	0	0
2.0	6.54250	.96260	.014622	.(3)19025	.(5)17385	.(7)11441	0	0
2.2	6.64968	.95553	.017307	.(3)27054	.(5)29786	.(7)23652	0	0
2.4	6.76466	.94805	.020116	.(3)37128	.(5)48422	.(7)45618	0	0
2.5	6.82491	.94419	.021557	.(3)42994	.(5)60693	.(7)61939	.(9)47390	0
2.6	6.88690	.94024	.023020	.(3)49443	.(5)75297	.(7)82972	.(9)68580	0
2.8	7.01589	.93213	.025995	.(3)64164	.(4)11271	.(6)14352	.(8)13722	0
3.0	7.15110	.92379	.029016	.(3)81422	.(4)16323	.(6)23767	.(8)26014	0
3.2	7.29204	.91526	.032062	.(2)10131	.(4)22964	.(6)37885	.(8)47040	0
3.4	7.43823	.90659	.035112	.(2)12389	.(4)31491	.(6)58389	.(8)81583	0
3.5	7.51315	.90221	.036632	.(2)13619	.(4)36557	.(6)71662	.(7)10593	0
3.6	7.58921	.89782	.038147	.(2)14917	.(4)42211	.(6)87331	.(7)13634	0
3.8	7.74456	.88900	.041152	.(2)17715	.(4)55439	.(5)12716	.(7)22040	0
4.0	7.90386	.88015	.044112	.(2)20777	.(4)71489	.(5)18073	.(7)34579	.(9)51546
4.2	8.06675	.87130	.047014	.(2)24097	.(4)90666	.(5)25132	.(7)52800	.(9)86517
4.4	8.23288	.86250	.049850	.(2)27664	.(3)11327	.(5)34259	.(7)78664	.(8)14102
4.5	8.31705	.85811	.051240	.(2)29536	.(3)12594	.(5)39724	.(7)95199	.(8)17822
4.6	8.40193	.85375	.052611	.(2)31465	.(3)13957	.(5)45861	.(6)11459	.(8)22379
4.8	8.57361	.84508	.055290	.(2)35488	.(3)16983	.(5)60381	.(6)16353	.(8)34653
5.0	8.74767	.83650	.057882	.(2)39717	.(3)20429	.(5)78295	.(6)22899	.(8)52463

c	$+d_{-2}$	$-d_{-4}$	$-d_{-6}/\rho$	$+d_{-8}/\rho$	$-d_{-10}/\rho$	$+d_{-12}/\rho$	$-d_{-14}/\rho$	$+d_{-16}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(2)13396	.(5)66790	.(9)74187	0	0	0	0	0
0.2	.(2)54331	.(3)10744	.(7)47690	.(11)75661	0	0	0	0
0.4	.022954	.(2)17561	.(5)31062	.(8)19683	0	0	0	0
0.5	.037330	.(2)43553	.(4)12002	.(7)11870	0	0	0	0
0.6	.056390	.(2)92040	.(4)36399	.(7)51771	0	0	0	0
0.8	.11262	.030472	.(3)21236	.(6)53514	0	0	0	0
1.0	.20209	.078619	.(3)84664	.(5)33190	.(8)92240	0	0	0
1.2	.33879	.17294	.(2)26459	.(4)14857	.(7)59288	0	0	0
1.4	.53811	.33843	.(2)69376	.(4)52689	.(6)28522	0	0	0
1.5	.66492	.45599	.010638	.(4)92426	.(6)57329	0	0	0
1.6	.81020	.60001	.015781	.(3)15542	.(5)10947	0	0	0
1.8	1.1478	.96804	.031589	.(3)39055	.(5)34660	0	0	0
2.0	1.5140	1.4179	.055891	.(3)84541	.(5)92174	0	0	0
2.2	1.8464	1.8833	.087734	.(2)15899	.(4)20861	.(6)19415	0	0
2.4	2.0861	2.2834	.12344	.(2)26335	.(4)40879	.(6)45106	0	0
2.5	2.1622	2.4404	.14128	.(2)32518	.(4)54597	.(6)65237	0	0
2.6	2.2096	2.5649	.15844	.(2)39211	.(4)70974	.(6)91536	0	0
2.8	2.2311	2.7212	.18955	.(2)53730	.(3)11201	.(5)16680	0	0
3.0	2.1830	2.7767	.21563	.(2)69241	.(3)16447	.(5)27983	0	0
3.2	2.0966	2.7645	.23695	.(2)85363	.(3)22889	.(5)44086	0	0
3.4	1.9939	2.7120	.25431	.010191	.(3)30589	.(5)66154	0	0
3.5	1.9411	2.6773	.26181	.011033	.(3)34940	.(5)79848	0	0
3.6	1.8885	2.6389	.26863	.011884	.(3)39638	.(5)95558	0	0
3.8	1.7872	2.5571	.28065	.013613	.(3)50120	.(4)13382	.(6)26577	0
4.0	1.6933	2.4738	.29090	.015374	.(3)62107	.(4)18257	.(6)39999	0
4.2	1.6073	2.3920	.29974	.017166	.(3)75678	.(4)24364	.(6)58576	0
4.4	1.5296	2.3142	.30750	.018985	.(3)90889	.(4)31892	.(6)83741	0
4.5	1.4937	2.2771	.31105	.019905	.(3)99133	.(4)36253	.(6)99315	0
4.6	1.4594	2.2409	.31436	.020827	.(2)10778	.(4)41036	.(5)11717	.(7)25783
4.8	1.3963	2.1727	.32045	.022685	.(2)12638	.(4)51998	.(5)16080	.(7)38372
5.0	1.3390	2.1088	.32584	.024553	.(2)14668	.(4)64974	.(5)21681	.(7)55947

c	$-A_{2,1}$	$+d_1$	$-d_3$	$+d_5$	$-d_7$	$+d_9$	$-d_{11}$	$+d_{13}$
0	12.	1.	0	0	0	0	0	0
0.1	12.00333	0.99981	.(4)53897	.(8)17131	0	0	0	0
0.2	12.01333	.99926	.(3)21144	.(7)26880	0	0	0	0
0.4	12.05328	.99705	.(3)84329	.(6)42933	0	0	0	0
0.5	12.08319	.99539	.(2)13148	.(5)10465	.(9)55219	0	0	0
0.6	12.11971	.99337	.(2)18883	.(5)21584	.(8)16397	0	0	0
0.8	12.21242	.98828	.(2)33343	.(5)67689	.(8)91381	0	0	0
1.0	12.33110	.98179	.(2)51649	.(4)16368	.(7)34497	0	0	0
1.2	12.47538	.97398	.(2)73595	.(4)33538	.(6)10170	0	0	0
1.4	12.64481	.96489	.(2)98935	.(4)61268	.(6)25263	.(9)74269	0	0
1.5	12.73878	.95989	.011279	.(4)80111	.(6)37808	.(8)12785	0	0
1.6	12.83884	.95459	.012739	.(3)10285	.(6)55323	.(8)21226	0	0
1.8	13.05688	.94318	.015866	.(3)16176	.(5)10998	.(8)53357	0	0
2.0	13.29825	.93072	.019242	.(3)24160	.(5)20249	.(7)12115	0	0
2.2	13.56223	.91732	.022831	.(3)34592	.(5)35020	.(7)25323	0	0
2.4	13.84802	.90305	.026599	.(3)47816	.(5)57499	.(7)49417	0	0
2.5	13.99885	.89563	.028538	.(3)55576	.(5)72441	.(7)67508	.(9)47188	0
2.6	14.15480	.88803	.030509	.(3)64152	.(5)90344	.(7)90995	.(9)68759	0
2.8	14.48169	.87235	.034526	.(3)83890	.(4)13670	.(6)15943	.(8)13955	0
3.0	14.82778	.85610	.038651	.(2)10728	.(4)20017	.(6)26752	.(8)26848	0
3.2	15.19213	.83939	.042741	.(2)13451	.(4)28479	.(6)43225	.(8)49289	0
3.4	15.57378	.82230	.046873	.(2)16576	.(4)39499	.(6)67541	.(8)86841	0
3.5	15.77080	.81365	.048931	.(2)18291	.(4)46116	.(6)83471	.(7)11360	0
3.6	15.97178	.80494	.050979	.(2)20109	.(4)53552	.(5)10243	.(7)14737	0
3.8	16.38516	.78738	.055033	.(2)24057	.(4)71136	.(5)15125	.(7)24202	0
4.0	16.81296	.76970	.059007	.(2)28417	.(4)92764	.(5)21798	.(7)38577	.(9)53233
4.2	17.25423	.75200	.062881	.(2)33183	.(3)11895	.(5)30733	.(7)59844	.(9)90904
4.4	17.70805	.73432	.066633	.(2)38341	.(3)15022	.(5)42470	.(7)90568	.(8)15074
4.5	17.93939	.72550	.068457	.(2)41064	.(3)16893	.(5)49880	.(6)11113	.(8)19331
4.6	18.17352	.71674	.070246	.(2)43876	.(3)18705	.(5)57621	.(6)13400	.(8)24334
4.8	18.64978	.69932	.073708	.(2)49766	.(3)22993	.(5)76869	.(6)19417	.(8)38322
5.0	19.13598	.68211	.077006	.(2)55988	.(3)27928	.(4)10096	.(6)27600	.(8)58991

c	d_{-1}	$+d_{-3}$	$+d_{-5}/\rho$	$-d_{-7}/\rho$	$+d_{-9}/\rho$	$-d_{-11}/\rho$	$+d_{-13}/\rho$	$-d_{-15}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(3)57057	.(6)95307	.(9)52932	0	0	0	0	0
0.2	.(2)22718	.(4)15281	.(7)33915	0	0	0	0	0
0.4	.(2)89178	.(3)24658	.(5)21807	.(8)24888	0	0	0	0
0.5	.013732	.(3)60583	.(5)83479	.(7)14875	0	0	0	0
0.6	.019413	.(2)12661	.(4)25035	.(7)64175	0	0	0	0
0.8	.032822	.(2)40820	.(3)14224	.(6)64662	0	0	0	0
1.0	.047707	.010225	.(3)55048	.(5)38979	.(7)15685	0	0	0
1.2	.061972	.021885	.(2)16736	.(4)16999	.(7)98323	0	0	0
1.4	.072644	.042095	.(2)43123	.(4)59349	.(6)46659	0	0	0
1.5	.075390	.056648	.(2)66031	.(3)10405	.(6)93718	0	0	0
1.6	.075563	.074995	.(2)98533	.(3)17618	.(5)18031	0	0	0
1.8	.064963	.12615	.020553	.(3)46230	.(5)59712	0	0	0
2.0	.032935	.20294	.039913	.(2)11009	.(4)17497	.(6)17778	0	0
2.2	-.031157	.31489	.073127	.(2)24221	.(4)46411	.(6)56918	0	0
2.4	-.14113	.47371	.12752	.(2)49850	.(3)11322	.(5)16485	0	0
2.5	-.21852	.57492	.16562	.(2)69941	.(3)17199	.(5)27138	0	0
2.6	-.31391	.69289	.21285	.(2)96768	.(3)25679	.(5)43766	0	0
2.8	-.56799	.98537	.34077	.017792	.(3)54492	.(4)10740	.(6)14704	0
3.0	-.91912	1.3592	.52302	.031016	.(2)10848	.(4)24468	.(6)38377	0
3.2	-1.3722	1.8098	.76696	.051162	.(2)20243	.(4)51772	.(6)92182	0
3.4	-1.9105	2.3118	1.0692	.079540	.(2)35311	.(3)10158	.(5)20368	0
3.5	-2.1983	2.5672	1.2365	.076860	.(2)45419	.(3)13818	.(5)29323	0
3.6	-2.4880	2.8151	1.4094	.11604	.(2)57374	.(3)18430	.(5)41321	0
3.8	-3.0353	3.2545	1.7514	.15847	.(2)86687	.(3)30894	.(5)76957	0
4.0	-3.4317	3.5725	2.0531	.20290	.012206	.(3)47980	.(4)13203	.(6)26778
4.2	-3.7822	3.7413	.24500	.24500	.016121	.(3)69547	.(4)21030	.(6)46912
4.4	-3.9289	3.7682	2.4291	.28159	.020165	.(3)94957	.(4)31405	.(6)76683
4.5	-3.9510	3.7378	2.4720	.29729	.022170	.(2)10890	.(4)37603	.(6)95908
4.6	-3.9449	3.6849	2.4976	.31125	.024146	.(2)12360	.(4)44513	.(5)11847
4.8	-3.8652	3.5290	2.5045	.33404	.027954	.(2)15490	.(4)60506	.(5)17482
5.0	-3.7256	3.3345	2.4685	.35093	.031555	.(2)18858	.(4)79598	.(5)24875

c	$+d_0$	$-d_2$	$+d_4$	$-d_6$	$+d_8$	$-d_{10}$	$+d_{12}$	$-d_{14}$
0	1.00000	0	0	0	0	0	0	0
0.1	0.99994	.(4)17633	.(9)33627	.(14)50718	0	0	0	0
0.2	.99976	.(4)70492	.(8)53761	.(12)32429	0	0	0	0
0.4	.99902	.(3)28131	.(7)85745	.(10)20678	.(14)38678	0	0	0
0.5	.99846	.(3)43878	.(6)20884	.(10)78663	.(13)22984	0	0	0
0.6	.99779	.(3)63050	.(6)43180	.(9)23409	.(13)98460	0	0	0
0.8	.99609	.(2)11148	.(5)13547	.(8)13040	.(12)95296	0	0	0
1.0	.99392	.(2)17299	.(5)32762	.(8)49200	.(11)57374	.(14)52712	0	0
1.2	.99130	.(2)24703	.(5)67160	.(7)14496	.(10)24310	.(13)32131	0	0
1.4	.98825	.(2)33293	.(4)12276	.(7)35982	.(10)82006	.(12)10839	0	0
1.5	.98657	.(2)38011	.(4)16056	.(7)53959	.(9)14105	.(12)29080	0	0
1.6	.98479	.(2)42998	.(4)20621	.(7)78739	.(9)23397	.(12)54846	0	0
1.8	.98094	.(2)53736	.(4)32462	.(6)15642	.(9)58707	.(11)17391	0	0
2.0	.97672	.(2)65420	.(4)48535	.(6)28777	.(8)13304	.(11)48578	0	0
2.2	.97216	.(2)77960	.(4)69580	.(6)49738	.(8)27754	.(10)12240	.(13)43457	0
2.4	.96729	.(2)91262	.(4)96325	.(6)81620	.(8)54054	.(10)28313	.(12)11945	0
2.5	.96474	.(2)98169	.(3)11205	.(5)10281	.(8)73771	.(10)41883	.(12)19158	0
2.6	.96213	.010523	.(3)12947	.(5)12819	.(8)99343	.(10)60937	.(12)30123	0
2.8	.95672	.011977	.(3)16965	.(5)19392	.(7)17373	.(9)12330	.(12)70564	0
3.0	.95107	.013480	.(3)21748	.(5)28394	.(7)29102	.(9)23651	.(11)15508	0
3.2	.94522	.015021	.(3)27345	.(5)40407	.(7)46946	.(9)43293	.(11)32233	0
3.4	.93918	.016592	.(3)33801	.(5)56069	.(7)73254	.(9)76045	.(11)63791	0
3.5	.93612	.017386	.(3)37363	.(5)65484	.(7)90477	.(9)99381	.(11)88220	0
3.6	.93298	.018184	.(3)41151	.(5)76073	.(6)11097	.(8)12875	.(10)12078	.(13)93136
3.8	.92669	.019792	.(3)49422	.(4)10116	.(6)16370	.(8)21096	.(10)21994	.(12)18861
4.0	.92027	.021405	.(3)58631	.(4)13211	.(6)23579	.(8)33554	.(10)38663	.(12)36660
4.2	.91376	.023019	.(3)68787	.(4)16971	.(6)33234	.(8)51957	.(10)65825	.(12)68662
4.4	.90719	.024627	.(3)79890	.(4)21478	.(6)45929	.(8)78512	.(9)10886	.(11)12433
4.5	.90389	.025427	.(3)85794	.(4)24036	.(6)53624	.(8)95696	.(9)13858	.(11)16536
4.6	.90057	.026223	.(3)91931	.(4)26812	.(6)62339	.(7)11602	.(9)17529	.(11)18750
4.8	.89394	.027802	.(2)10490	.(4)33055	.(6)83227	.(7)16797	.(9)27547	.(11)37262
5.0	.88728	.029361	.(2)11876	.(4)40285	.(5)10944	.(7)23865	.(9)42330	.(11)61966

c	d_{-2}	d_{-4}	d_{-6}	d_{-8}/ρ	d_{-10}/ρ	d_{-12}/ρ	d_{-14}/ρ
0	0	0	0	0	0	0	0
0.1	.(3)71238	.(5)23869	.(8)26501	.(12)94629	0	0	0
0.2	.(2)33845	.(4)46083	.(6)20421	.(10)29206	0	0	0
0.4	.012848	.(3)74747	.(4)13133	.(8)74804	0	0	0
0.5	.019590	.(2)18773	.(4)51199	.(7)45484	0	0	0
0.6	.026915	.(2)39787	.(3)15502	.(6)19787	-.(9)14342	0	0
0.8	.041315	.013288	.(3)90221	.(5)20359	-.(8)26165	0	0
1.0	.049056	.034799	.(2)36004	.(4)12604	-.(7)25225	0	0
1.2	.037639	.078543	.011359	.(4)56767	-.(6)16293	0	0
1.4	-.0143	.161	.0306	.(3)206	-.(6)801	0	0
1.5	-.049933	.16970	.036389	.(3)27970	-.(5)12450	0	0
1.6	-.14273	.30756	.073604	.(3)64435	-.(5)32536	.(7)14041	-.(10)47259
1.8	-.40912	.56251	.16353	.(2)17761	-.(4)11280	.(7)61383	-.(9)26085
2.0	-.92276	1.0019	.34418	.(2)45405	-.(4)35353	.(6)23654	-.(8)12376
2.2	-1.8974	1.7746	.70439	.011092	-.(3)10371	.(6)83583	-.(8)52754
2.4	-3.8280	3.2273	1.4530	.026778	-.(3)29548	.(5)28200	-.(7)21112
2.5	-5.5324	4.4765	2.1337	.042295	-.(3)50418	.(5)52075	-.(7)42230
2.6	-8.2551	6.4146	3.2405	.068849	-.(3)88365	.(5)98448	-.(7)86192
2.8	-24.105	17.731	9.8289	.20042	-.(2)29547	.(4)37958	-.(6)38393
....	∞	∞	∞	∞	∞	∞	∞
3.0	143.33	-100.54	-60.760	-1.6532	.027644	-.(3)40591	.(5)46934
3.2	25.968	-17.518	-11.434	-.34666	.(2)65362	-.(3)10829	.(5)14183
3.4	16.529	-10.792	-7.5455	-.25272	.(2)53182	-.(4)98778	.(5)14536
3.5	14.441	-9.2938	-6.7074	-.23542	.(2)52189	-.(3)10235	.(5)15921
3.6	12.919	-8.2034	-6.1048	-.22409	.(2)51477	-.(3)10641	.(5)17467
3.8	11.098	-6.8806	-5.4101	-.21625	.(2)55473	-.(3)12678	.(5)23064
4.0	9.8761	-5.9966	-4.9582	-.21444	.(2)60412	-.(3)15176	.(5)30420
4.2	8.9965	-5.3629	-4.6412	-.21597	.(2)65911	-.(3)18102	.(5)39770
4.4	8.3155	-4.8767	-4.3993	-.21915	.(2)72385	-.(3)21628	.(5)51828
4.5	8.0258	-4.6717	-4.2968	-.22107	.(2)75833	-.(3)23593	.(5)58948
4.6	7.7620	-4.4862	-4.2033	-.22312	.(2)79399	-.(3)25695	.(5)66867
4.8	7.2966	-4.1621	-4.0374	-.22743	.(2)86827	-.(3)30268	.(5)85192
5.0	6.8962	-3.8873	-3.8928	-.23181	.(2)94593	-.(3)35481	.(4)10760

OBLATE SEPARATION CONSTANTS

c	$B_{0,0}$	$B_{0,1}$	$B_{0,2}$	$B_{0,3}$	$B_{1,0}$	$B_{1,1}$	$B_{1,2}$	$B_{2,1}$	$B_{3,0}$
0	0	-2.00000	-6.00000	-12.00000	-2.00000	-6.00000	-12.00000	-12.00000	-12.00000
0.1	.00334	-1.99400	-5.99476	-11.99489	-1.99800	-5.99571	-11.99533	-11.99667	-11.99889
0.2	.01336	-1.97599	-5.97906	-11.97956	-1.99199	-5.98285	-11.98129	-11.98667	-11.99555
0.4	.05372	-1.90383	-5.91645	-11.91831	-1.96788	-5.93133	-11.92537	-11.94661	-11.98220
0.5	.08427	-1.84957	-5.86969	-11.87243	-1.94971	-5.89261	-11.88342	-11.91653	-11.97216
0.6	.12194	-1.78311	-5.81277	-11.81642	-1.92740	-5.84521	-11.83218	-11.87971	-11.95987
0.8	.21952	-1.61321	-5.66904	-11.67422	-1.87010	-5.72412	-11.70192	-11.78575	-11.92848
1.0	.34860	-1.39321	-5.48680	-11.49212	-1.79531	-5.56753	-11.53482	-11.66441	-11.88788
1.2	.51205	-1.12197	-5.26811	-11.27064	-1.70215	-5.37476	-11.33119	-11.51531	-11.83789
1.4	.71352	-0.79815	-5.01563	-11.01039	-1.58949	-5.14497	-11.09150	-11.33795	-11.77829
1.5	.82987	-.61604	-4.87773	-10.86594	-1.52542	-5.01589	-10.95831	-11.23849	-11.74479
1.6	.95746	-.42019	-4.73267	-10.71204	-1.45591	-4.87717	-10.81635	-11.13175	-11.70879
1.8	1.24914	.01366	-4.42316	-10.37629	-1.29965	-4.57020	-10.50657	-10.89601	-11.62906
2.0	1.59449	.50524	-4.09151	-10.00387	-1.11855	-4.22275	-10.16325	-10.62995	-11.53870
2.2	1.99992	1.05654	-3.74241	-9.59543	-0.91004	-3.83335	-9.78777	-10.33268	-11.43725
2.4	2.47189	1.66959	-3.38044	-9.15158	--.67100	-3.40037	-9.38194	-10.00317	-11.32417
2.5	2.73476	1.99990	-3.19590	-8.91653	-.53890	-3.16699	-9.16831	-9.82598	-11.26307
2.6	3.01649	2.34642	-3.00956	-8.67280	-.39773	-2.92204	-8.94797	-9.64030	-11.19882
2.8	3.63894	3.08904	-2.63268	-8.15939	-.08591	-2.39644	-8.48860	-9.24284	-11.06048
3.0	4.34329	3.89940	-2.25127	-7.61147	.26942	-1.82154	-8.00707	-8.80940	-10.90830
3.2	5.13226	4.77932	-1.86518	-7.02888	.67391	-1.19520	-7.50713	-8.33850	-10.74129
3.4	6.00733	5.73048	-1.47273	-6.41120	1.13368	-0.51517	-6.99288	-7.82852	-10.55833
3.5	6.47733	6.23325	-1.27321	-6.08900	1.38633	-.15432	-6.73173	-7.55833	-10.46046
3.6	6.96898	6.75440	-1.07087	-5.75774	1.65523	.22081	-6.46860	-7.27772	-10.35809
3.8	8.01700	7.85242	-0.65545	-5.06747	2.24502	1.01505	-5.93842	-6.68423	-10.13904
4.0	9.15080	9.02571	-.22141	-4.33908	2.90920	1.86986	-5.40590	-6.04608	-9.89941
4.2	10.36957	10.27526	.23696	-3.57098	3.65313	2.78749	-4.87366	-5.36120	-9.63715
4.4	11.67250	11.60189	.72602	-2.76131	4.48108	3.77011	-4.34298	-4.62740	-9.34982
4.5	12.35528	12.29433	.98420	-2.34023	4.92757	4.28644	-4.07821	-4.24145	-9.19593
4.6	13.05882	13.00628	1.25268	-1.90796	5.39610	4.81978	-3.81362	-3.84244	-9.03470
4.8	14.52786	14.48898	1.82429	-0.60929	6.39997	5.93843	-3.28382	-3.00402	-8.68856
5.0	16.07904	16.05041	2.44860	-.06093	7.49339	7.12783	-2.75037	-2.10982	-8.30773

c	f_0	f_2	f_4	f_6	f_8	f_{10}	f_{12}	f_{14}
0	1.0000	0	0	0	0	0	0	0
0.1	1.0006	.(2)11121	.(6)19066	0	0	0	00	0
0.2	1.0021	.(2)44595	.(5)30590	0	0	0	0	0
0.4	1.0090	.018028	.(4)49517	.(7)57199	0	0	0	0
0.5	1.0142	.028393	.(3)12194	.(6)22017	0	0	0	0
0.6	1.0205	.041283	.(3)25555	.(6)66471	0	0	0	0
0.8	1.0373	.075232	.(3)82977	.(5)38409	.(8)98187	0	0	0
1.0	1.0599	.12138	.(2)20975	.(4)15190	.(7)60715	0	0	0
1.2	1.0892	.18182	.(2)45385	.(4)47395	.(6)27302	0	0	0
1.4	1.1264	.25940	.(2)88418	.(3)12587	.(6)98759	.(8)49159	0	0
1.5	1.1484	.30573	.011983	.(3)19595	.(5)17660	.(7)10093	0	0
1.6	1.1730	.35783	.015982	.(3)29758	.(5)30526	.(7)19854	0	0
1.8	1.2309	.48192	.027325	.(3)64475	.(5)83766	.(7)68986	0	0
2.0	1.3025	.63781	.044756	.(2)13051	.(4)20944	.(6)21302	.(8)14967	0
2.2	1.3908	.83313	.070853	.(2)25013	.(4)48586	.(6)59804	.(8)50848	0
2.4	1.4991	1.0773	.10908	.(2)45829	.(3)10593	.(5)15517	.(7)15700	0
2.5	1.5621	1.2211	.13415	.(2)61137	.(3)15331	.(5)24363	.(7)26745	0
2.6	1.6318	1.3816	.16402	.(2)80815	.(3)21912	.(5)37655	.(7)44703	0
2.8	1.7933	1.7595	.24162	.013785	.(3)43305	.(5)86248	.(6)11869	0
3.0	1.9893	2.2268	.34948	.022832	.(3)82211	.(4)18776	.(6)29640	.(8)34277
3.2	2.2257	2.8022	.49721	.036831	.(2)15057	.(4)39070	.(6)70101	.(8)92164
3.4	2.5099	3.5075	.69682	.058014	.(2)26673	.(4)77990	.(5)15776	.(7)23391
3.5	2.6723	3.9169	.82063	.072213	.(2)35164	.(3)10884	.(5)23313	.(7)36607
3.6	2.8499	4.3688	.96329	.089428	.(2)45990	.(3)15042	.(5)34059	.(7)56545
3.8	3.2556	5.4171	1.3152	.13518	.(2)77158	.(3)28045	.(5)70618	.(6)13044
4.0	3.7382	6.6894	1.7755	.20076	.012640	.(3)50754	.(4)14130	.(6)28871
4.2	4.3116	8.2301	2.3728	.29342	.020266	.(3)89414	.(4)27378	.(6)61541
4.4	4.9920	10.093	3.1423	.42270	.031867	.(2)15373	.(4)51523	.(5)12688
4.5	5.3782	11.165	3.6049	.50482	.039691	.(2)19988	.(4)69969	.(5)20904
4.6	5.7987	12.343	4.1275	.60104	.049229	.(2)25852	.(4)94420	.(5)25355
4.8	6.7549	15.057	5.3822	.84464	.074843	.(2)42606	.(3)16890	.(5)49263
5.0	7.8884	18.330	6.9727	1.1744	.11214	.(2)68943	.(3)29555	.(5)93286

OBLATE SPHEROIDAL FUNCTIONS $m=0$, $\ell=0$

c	f_{-2}/ρ	f_{-4}/ρ	f_{-6}/ρ	f_{-8}/ρ	f_{-10}/ρ	f_{-12}/ρ	f_{-14}/ρ	f_{-16}/ρ
0	0	0	0	0	0	0	0	0
0.1	.(2)50094	.(5)16701	.(9)17674	.(14)92697	0	0	0	0
0.2	.020150	.(4)26882	.(7)11381	.(11)23880	0	0	0	0
0.4	.082469	.(3)44087	.(6)74712	.(9)62725	0	0	0	0
0.5	.13110	.(2)10965	.(5)29048	.(8)38116	.(11)29909	0	0	0
0.6	.19284	.(2)23262	.(5)88792	.(7)16783	.(10)18967	0	0	0
0.8	.36215	.(2)77963	.(4)52984	.(6)17817	.(9)35815	.(12)47909	0	0
1.0	.60816	.020554	.(3)21865	.(5)11499	.(8)36137	.(11)75562	0	0
1.2	.95875	.046913	.(3)72006	.(5)54589	.(7)24719	.(10)74463	0	0
1.4	1.4573	.097621	.(2)20438	.(4)21112	.(6)13021	.(9)53413	.(11)15629	0
1.5	1.7819	.13743	.(2)33065	.(4)39231	.(6)27786	.(8)13087	.(11)43966	0
1.6	2.1716	.19112	.(2)52370	.(4)70736	.(6)57019	.(8)30562	.(10)11684	0
1.8	3.2087	.35940	.012486	.(3)21365	.(5)21809	.(7)14800	.(10)71631	0
2.0	4.7398	.65858	.028292	.(3)59809	.(5)75405	.(7)63194	.(9)37766	.(11)1691
2.2	7.0416	1.1880	.061808	.(2)15816	.(4)24133	.(6)24476	.(8)17701	.(11)9591
2.4	10.565	2.1249	.13158	.(2)40069	.(4)72756	.(6)87809	.(8)75570	.(10)4843
2.5	13.002	2.8374	.19059	.(2)62961	.(3)12403	.(5)16240	.(7)15164	.(9)10609
2.6	16.054	3.7872	.27499	.(2)98216	.(3)20921	.(5)29624	.(7)29915	.(9)22633
2.8	24.736	6.7461	.56694	.023455	.(3)57897	.(5)95025	.(6)11124	.(9)97576
3.0	38.654	12.029	1.1567	.054827	.(2)15516	.(4)29209	.(6)39227	.(8)39479
3.2	61.219	21.481	2.3394	.12583	.(2)40650	.(4)86959	.(5)13276	.(7)15191
3.4	98.136	38.364	4.6888	.28373	.010273	.(3)24771	.(5)42643	.(7)55037
3.5	124.76	51.385	6.6321	.42443	.016264	.(3)41520	.(5)75692	.(6)10347
3.6	158.99	68.749	9.3520	.63181	.025577	.(3)69013	.(4)13301	.(6)19225
3.8	259.94	123.09	18.498	1.3856	.062303	.(2)18689	.(4)40067	.(6)64443
4.0	428.25	220.32	36.337	2.9990	.14887	.(2)49357	.(3)11703	.(5)20826
4.2	710.02	394.09	70.892	6.4105	.34941	.012735	.(3)32878	.(5)64398
4.4	1183.7	704.33	137.43	13.545	.80665	.032166	.(3)91881	.(4)19715
4.5	1531.0	941.23	190.90	19.608	1.2185	.050738	.(2)15140	.(4)33948
4.6	1982.3	1257.4	264.78	28.310	1.8337	.079648	.(2)24803	.(4)58053
4.8	3332.8	2242.1	507.16	58.570	4.1090	.19361	.(2)65470	.(3)16649
5.0	5624.1	3994.6	966.46	120.07	9.0889	.46284	.016932	.(3)46612

c	f_1	f_3	f_5	f_7	f_9	f_{11}	f_{13}	f_{15}
0	1.0000	0	0	0	0	0	0	0
0.1	1.0006	.(3)40020	.(7)45373	.(11)24678	0	0	0	0
0.2	1.0024	.(2)16033	.(5)72700	.(8)15815	0	0	0	0
0.4	1.0097	.(2)64526	.(4)11699	.(7)10178	.(11)52240	0	0	0
0.5	1.0151	.010129	.(4)28684	.(7)40001	.(10)31263	0	0	0
0.6	1.0219	.014667	.(4)59792	.(6)11700	.(9)13508	0	0	0
0.8	1.0393	.026451	.(3)19150	.(6)66581	.(8)13662	0	0	0
1.0	1.0623	.042097	.(3)47557	.(5)25817	.(8)82736	0	0	0
1.2	1.0911	.061998	.(2)10068	.(5)78634	.(7)36267	.(9)11062	0	0
1.4	1.1264	.086655	.(2)19114	.(4)20296	.(6)12732	.(9)52833	0	0
1.5	1.1467	.10096	.(2)25533	.(4)31102	.(6)22389	.(8)10662	0	0
1.6	1.1689	.11670	.(2)33535	.(4)46444	.(6)38022	.(8)20595	0	0
1.8	1.2192	.15290	.(2)55439	.(4)97016	.(5)10042	.(8)68792	0	0
2.0	1.2783	.19619	.(2)87509	.(3)18869	.(5)24082	.(7)20351	.(9)12236	0
2.2	1.3474	.24771	.013313	.(3)34656	.(5)53444	.(7)54593	.(9)39688	0
2.4	1.4277	.30881	.019657	.(3)60732	.(4)11128	.(6)13513	.(8)11681	0
2.5	1.4725	.34345	.023659	.(3)79204	.(4)15732	.(6)20715	.(8)19860	0
2.6	1.5208	.38111	.028315	.(2)10237	.(4)21970	.(6)31268	.(8)31692	0
2.8	1.6284	.46653	.039950	.(2)16693	.(4)41459	.(6)68332	.(8)80235	0
3.0	1.7528	.56735	.055384	.(2)26463	.(4)75274	.(5)14218	.(7)19141	.(9)19300
3.2	1.8962	.68628	.075635	.(2)40941	.(3)13212	.(5)28339	.(7)43348	.(9)49675
3.4	2.0616	.82651	.10196	.(2)62003	.(3)22519	.(5)54412	.(7)93813	.(8)12122
3.5	2.1535	.90577	.11786	.(2)75757	.(3)29108	.(5)74447	.(6)13590	.(8)18596
3.6	2.2522	.99183	.13589	.(2)92155	.(3)37396	.(4)10107	.(6)19501	.(8)28212
3.8	2.4719	1.1867	.17932	.013471	.(3)60681	.(4)18225	.(6)39108	.(8)62944
4.0	2.7250	1.4164	.23456	.019402	.(3)96445	.(4)32005	.(6)75939	.(7)13521
4.2	3.0168	1.6872	.30444	.027574	.(2)15045	.(4)54876	.(5)14323	.(7)28065
4.4	3.3532	2.0065	.39239	.038721	.(2)23078	.(4)92074	.(5)26310	.(7)56472
4.5	3.5404	2.1869	.44442	.045696	.(2)28415	.(3)11837	.(5)35335	.(7)79247
4.6	3.7414	2.3829	.50260	.053788	.(2)34859	.(3)15147	.(5)47183	.(6)11046
4.8	4.1896	2.8268	.64019	.073982	.(2)51924	.(3)24473	.(5)82773	.(6)21052
5.0	4.7072	3.3504	.81133	.10085	.(2)76360	.(3)38894	.(4)14231	.(6)39180

c	f_{-1}/ρ	f_{-3}/ρ	f_{-5}/ρ	f_{-7}/ρ	f_{-9}/ρ	f_{-11}/ρ	f_{-13}/ρ	f_{-15}/ρ
0	0	0	0	0	0	0	0	0
0.1	.(2)16699	.(5)27826	.(9)53000	0	0	0	0	0
0.2	.(2)67184	.(4)44755	.(7)34029	.(10)10330	0	0	0	0
0.4	.027506	.(3)73124	.(5)22267	.(8)26979	0	0	0	0
0.5	.043734	.(2)18135	.(5)86245	.(7)16324	.(10)16737	0	0	0
0.6	.064338	.(2)38336	.(4)26237	.(7)71493	.(9)10554	0	0	0
0.8	.12079	.012725	.(3)15460	.(6)74835	.(8)19631	0	0	0
1.0	.20253	.033098	.(3)62700	.(5)47378	.(7)19408	.(10)50024	0	0
1.2	.31805	.074175	.(2)20182	.(4)21934	.(6)15103	.(9)56030	0	0
1.4	.47990	.15069	.(2)55629	.(4)82164	.(6)65865	.(8)33239	0	0
1.5	.58367	.20909	.(2)88447	.(3)14983	.(5)13781	.(8)79812	0	0
1.6	.70654	.28605	.013740	.(3)26458	.(5)27672	.(7)18227	.(10)82648	0
1.8	1.0251	.51746	.031315	.(3)76152	.(4)10067	.(7)83856	.(9)48094	0
2.0	1.4759	.90408	.067187	.(2)20119	.(4)32787	.(6)33682	.(8)23832	0
2.2	2.1189	1.5402	.13765	.(2)49723	.(4)97874	.(5)12152	.(7)10395	.(10)6502
2.4	3.0440	2.5766	.27211	.011657	.(3)27251	.(5)40211	.(7)40897	.(9)40321
2.5	3.6526	3.3156	.37845	.017559	.(3)44487	.(5)71174	.(7)78504	.(9)63337
2.6	4.3877	4.2551	.52314	.026197	.(3)71703	.(4)12398	.(6)14782	.(8)12893
2.8	6.3577	6.9643	.98408	.056894	.(2)18011	.(4)36079	.(6)49827	.(8)50357
3.0	9.2733	11.332	1.8197	.12017	.(2)43537	.(4)99839	.(5)15805	.(7)18317
3.2	13.629	18.376	3.3200	.24803	.010189	.(3)26526	.(5)47701	.(7)62819
3.4	20.194	29.748	5.9935	.50230	.023208	.(3)68029	.(4)13786	.(6)20467
3.5	24.659	37.842	8.0266	.71036	.034708	.(2)10767	.(4)23097	.(6)36311
3.6	30.177	48.144	10.730	1.0012	.051645	.(2)16925	.(4)38374	.(6)64773
3.8	45.481	77.971	19.085	1.9693	.11266	.(2)41009	.(3)10337	.(5)19109
4.0	69.131	126.46	33.773	3.8299	.24156	.(2)97102	.(3)27055	.(5)55312
4.2	105.95	205.54	59.529	7.3772	.51021	.022530	.(3)69026	.(4)15527
4.4	163.67	334.83	104.59	14.093	1.0634	.051335	.(2)17213	.(4)42400
4.5	204.02	427.79	138.52	19.425	1.5285	.077016	.(2)26970	.(4)69410
4.6	254.76	546.89	183.33	26.728	2.1908	.11510	.(2)42052	.(3)11295
4.8	402.14	901.81	322.91	50.197	4.4503	.25343	.010049	.(3)29316
5.0	630.31	1471.0	560.28	94.428	9.0201	.55469	.023784	.(3)75078

c	$-f_0$	$+f_2$	$+f_4$	$+f_6$	$+f_8$	$+f_{10}$	$+f_{12}$	$+f_{14}$
0	0	1.0000	0	0	0	0	0	0
0.1	.(3)22223	0.99974	.(3)24483	.(7)20608	0	0	0	0
0.2	.(3)88908	.99896	.(3)97852	.(6)32889	0	0	0	0
0.4	.(2)35586	.99581	.(2)39012	.(5)52537	.(8)36574	0	0	0
0.5	.(2)55626	.99343	.(2)60807	.(4)12794	.(7)13917	0	0	0
0.6	.(2)80138	.99050	.(2)87297	.(4)26449	.(7)41427	0	0	0
0.8	.014260	.98298	.015400	.(4)82943	.(6)23094	0	0	0
1.0	.022294	.97316	.023824	.(3)20049	.(6)87222	0	0	0
1.2	.032099	.96097	.033892	.(3)41074	.(5)25733	.(8)99270	0	0
1.4	.043627	.94639	.045476	.(3)75038	.(5)63997	.(7)33606	0	0
1.5	.050013	.93821	.051794	.(3)98135	.(5)96089	.(7)57929	0	0
1.6	.056795	.92946	.058442	.(2)12603	.(4)14043	.(7)96335	0	0
1.8	.071463	.91035	.072663	.(2)19852	.(4)28010	.(6)24327	0	0
2.0	.087433	.88933	.088035	.(2)29740	.(4)51846	.(6)55616	0	0
2.2	.10445	.86685	.10451	.(2)42814	.(4)90419	.(5)11745	.(7)10369	0
2.4	.12222	.84349	.12210	.(2)59712	.(3)15032	.(5)23260	.(7)24455	0
2.5	.13128	.83168	.13134	.(2)69831	.(3)19094	.(5)32078	.(7)36610	0
2.6	.14042	.81991	.14092	.(2)81212	.(3)24044	.(5)43723	.(7)53998	0
2.8	.15877	.79679	.16121	.010830	.(3)37286	.(5)78763	.(6)11294	0
3.0	.17705	.77478	.18329	.014224	.(3)56399	.(4)13704	.(6)22589	0
3.2	.19512	.75436	.20760	.018468	.(3)83639	.(4)23179	.(6)43541	0
3.4	.21296	.73590	.23471	.023776	.(2)12210	.(4)38303	.(6)81379	.(7)12511
3.5	.22182	.72746	.24951	.026911	.(2)14681	.(4)48875	.(5)11015	.(7)17959
3.6	.23065	.71957	.26527	.030422	.(2)17602	.(4)62097	.(5)14822	.(7)25585
3.8	.24835	.70544	.30006	.038759	.(2)25126	.(4)99092	.(5)26414	.(7)50882
4.0	.26632	.69345	.34000	.049243	.(2)35580	.(3)15605	.(5)46285	.(7)98962
4.2	.28487	.68348	.38618	.062455	.(2)50067	.(3)24302	.(5)79532	.(6)18782
4.4	.30439	.67538	.43995	.079147	.(2)70093	.(3)37488	.(4)13501	.(6)35056
4.5	.31465	.67194	.47014	.089086	.(2)82801	.(3)46413	.(4)17507	.(6)47593
4.6	.32533	.66888	.50286	.10027	.(2)97719	.(3)57353	.(4)22637	.(6)64364
4.8	.34823	.66377	.57684	.12707	.013576	.(3)87113	.(4)37539	.(5)11643
5.0	.37369	.65972	.66416	.16110	.018804	.(2)13144	.(4)61624	.(5)20777

OBLATE SPHEROIDAL FUNCTIONS m=0, ℓ=2

c	f_{-2}/ρ	f_{-4}/ρ	f_{-6}/ρ	f_{-8}/ρ	f_{-10}/ρ	f_{-12}/ρ	f_{-14}/ρ	f_{-16}/ρ
0	0	0	0	0	0	0	0	0
0.1	.(6)55547	0	0	0	0	0	0	0
0.2	.(5)88840	.(7)23689	0	0	0	0	0	0
0.4	.(3)14189	.(5)15131	.(8)32019	0	0	0	0	0
0.5	.(3)34590	.(5)57627	.(7)19053	0	0	0	0	0
0.6	.(3)71583	.(4)17172	.(7)81751	0	0	0	0	0
0.8	.(2)22491	.(4)95917	.(6)81174	.(8)30511	0	0	0	0
1.0	.(2)54399	.(3)36270	.(5)47960	.(7)28167	0	0	0	0
1.2	.011122	.(2)10694	.(4)20368	.(6)17226	0	0	0	0
1.4	.020188	.(2)26501	.(4)68731	.(6)79137	.(8)52137	0	0	0
1.5	.026255	.(2)39655	.(3)11811	.(5)15615	.(7)11811	0	0	0
1.6	.033468	.(2)57682	.(3)19559	.(5)29426	.(7)25327	0	0	0
1.8	.051575	.011343	.(3)48756	.(5)92902	.(6)10124	0	0	0
2.0	.074729	.020540	.(2)10927	.(4)25732	.(6)34640	.(8)30342	0	0
2.2	.10260	.034723	.(2)22429	.(4)64011	.(5)10436	.(7)11068	0	0
2.4	.13425	.055375	.(2)42774	.(3)14560	.(5)28285	.(7)35726	0	0
2.5	.15103	.068608	.(2)57675	.(3)21330	.(5)44998	.(7)61701	0	0
2.6	.16816	.084039	.(2)76669	.(3)30715	.(5)70144	.(6)10409	.(8)10872	0
2.8	.20245	.12230	.013044	.(3)60822	.(4)16143	.(6)27820	.(8)33731	0
3.0	.23505	.17185	.021250	.(2)11424	.(4)34895	.(6)69149	.(8)96360	0
3.2	.26394	.23469	.033411	.(2)20544	.(4)71610	.(5)16177	.(7)25684	0
3.4	.28724	.31338	.051059	.(2)35656	.(3)14079	.(5)35985	.(7)64600	0
3.5	.29622	.35970	.062578	.(2)46462	.(3)19477	.(5)52818	.(6)10057	.(8)14205
3.6	.30310	.41139	.076333	.(2)60167	.(3)26737	.(5)76807	.(6)15486	.(8)23157
3.8	.30950	.53359	.11227	.(2)99336	.(3)49392	.(4)15853	.(6)35680	.(8)59531
4.0	.30396	.68679	.16324	.016135	.(3)89299	.(4)31851	.(6)79598	.(7)14737
4.2	.28297	.88067	.23570	.025907	.(2)15884	.(4)61587	.(5)17006	.(7)34769
4.4	.24140	1.1290	.33921	.041293	.(2)27927	.(3)12129	.(5)36839	.(7)82793
4.5	.21059	1.2794	.40688	.052049	.(2)36913	.(3)16796	.(5)53422	.(6)12568
4.6	.17143	1.4515	.48824	.065573	.(2)48720	.(3)23203	.(5)77202	.(6)18995
4.8	.61090	1.8766	.70488	.10407	.(2)84628	.(3)44029	.(4)15988	.(6)42899
5.0	.10814	2.4461	1.0233	.16557	.014684	.(3)83161	.(4)32839	.(6)95760

c	$-f_1$	f_3	f_5	f_7	f_9	f_{11}	f_{13}	f_{15}
0	0	1.00000	0	0	0	0	0	0
0.1	.(3)17143	1.00011	.(3)17638	.(7)11774	0	0	0	0
0.2	.(3)68576	1.00043	.(3)70576	.(6)18844	.(10)27285	0	0	0
0.4	.(2)27436	1.00170	.(2)28265	.(5)30187	.(8)17484	0	0	0
0.5	.(2)42875	1.00266	.(2)44205	.(5)73766	.(8)66755	0	0	0
0.6	.(2)61750	1.00383	.(2)63728	.(4)15313	.(7)19955	0	0	0
0.8	.010982	1.00681	.011363	.(4)48539	.(6)11244	.(9)16533	0	0
1.0	.017167	1.01066	.017823	.(3)11895	.(6)43057	.(9)98915	0	0
1.2	.024733	1.01539	.025787	.(3)24785	.(5)12918	.(8)42735	0	0
1.4	.033678	1.02101	.035303	.(3)46185	.(5)32766	.(7)14754	0	0
1.5	.038668	1.02416	.040659	.(3)61250	.(5)49885	.(7)25785	0	0
1.6	.044003	1.02755	.046426	.(3)79340	.(5)73523	.(7)43240	0	0
1.8	.055706	1.03506	.059229	.(2)12813	.(4)15029	.(6)11187	.(9)57848	0
2.0	.068781	1.04357	.073801	.(2)19717	.(4)28556	.(6)26245	.(8)16755	0
2.2	.083225	1.05315	.090249	.(2)29187	.(4)51160	.(6)56901	.(8)43958	0
2.4	.099035	1.06389	.10871	.(2)41864	.(4)87353	.(5)11564	.(7)10633	0
2.5	.10745	1.06973	.11874	.(2)49635	.(3)11240	.(5)16420	.(7)16385	0
2.6	.11621	1.07589	.12933	.(2)58499	.(3)14331	.(5)22271	.(7)24036	0
2.8	.13476	1.08926	.15232	.(2)79980	.(3)22734	.(5)40986	.(7)51312	0
3.0	.15471	1.10414	.17790	.010736	.(3)35051	.(5)72564	.(6)10431	0
3.2	.17608	1.12070	.20634	.014187	.(3)52736	.(4)12427	.(6)20327	.(8)24501
3.4	.19894	1.13911	.23798	.018501	,(3)77695	.(4)20678	.(6)38200	.(8)51991
3.5	.21095	1.14975	.25512	.021035	.(3)93647	.(4)26417	.(6)51726	.(8)74608
3.6	.22337	1.15958	.27320	.023853	.(2)11240	.(4)33553	.(6)69518	.(7)10609
3.8	.24948	1.18234	.31243	.030455	.(2)16005	.(4)53263	.(5)12300	.(7)20920
4.0	.27744	1.20763	.35622	.038557	.(2)22474	.(4)82923	.(5)21226	.(7)40011
4.2	.30743	1.23573	.40518	.048463	.(2)31177	.(3)12690	.(5)35829	.(7)74476
4.4	.33972	1.26694	.46001	.060534	.(2)42788	.(3)19127	.(5)59290	.(6)13529
4.5	.35681	1.28380	.48988	.067514	.(2)49944	.(3)23359	.(5)75754	.(6)18083
4.6	.37459	1.30157	.52153	.075204	.(2)58166	.(3)28436	.(5)96381	.(6)24043
4.8	.41242	1.33997	.59071	.092991	.(2)78402	.(3)41760	.(4)15418	.(6)41885
5.0	.45365	1.38252	.66864	.11452	.010488	.(3)60650	.(4)24305	.(6)71657

OBLATE SPHEROIDAL FUNCTIONS $m=0, \quad \ell=3$

c	$+f_{-1}/\rho$	$-f_{-3}/\rho$	$-f_{-5}/\rho$	$-f_{-7}/\rho$	$-f_{-9}/\rho$	$-f_{-11}/\rho$	$-f_{-13}/\rho$	$-f_{-15}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(7)47626	.(10)52915	0	0	0	0	0	0
0.2	.(6)76238	.(8)33878	0	0	0	0	0	0
0.4	.(4)12223	.(6)21722	0	0	0	0	0	0
0.5	.(4)29888	.(6)82974	0	0	0	0	0	0
0.6	.(4)62091	.(5)24814	.(7)38278	0	0	0	0	0
0.8	.(3)19719	.(4)13993	.(6)38374	0	0	0	0	0
1.0	.(3)48451	.(4)53620	.(5)22981	.(7)23210	0	0	0	0
1.2	.(2)10129	.(3)16093	.(5)99376	.(6)14453	0	0	0	0
1.4	.(2)18956	.(3)40819	.(4)34336	.(6)67979	0	0	0	0
1.5	.(2)31925	.(3)78685	.(4)76032	.(5)17282	0	0	0	0
1.6	.(2)32738	.(3)91495	.(3)10067	.(5)26038	0	0	0	0
1.8	.(2)53221	.(2)18664	.(3)26044	.(5)85284	0	0	0	0
2.0	.(2)82553	.(2)35330	.(3)61053	.(4)24692	.(6)47290	0	0	0
2.2	.012342	.(2)62967	.(2)13221	.(4)64752	.(5)15010	0	0	0
2.4	.017915	.010675	.(2)26819	.(3)15648	.(5)43185	0	0	0
2.5	.021387	.013676	.(2)37404	.(3)23694	.(5)70972	0	0	0
2.6	.025398	.017353	.(2)51523	.(3)35367	.(4)11461	.(6)22047	0	0
2.8	.035332	.027218	.(2)94540	.(3)75289	.(4)28315	.(6)63192	0	0
3.0	.048419	.041405	.016685	.(2)15282	.(4)66027	.(5)16923	0	0
3.2	.065576	.061338	.028484	.(2)29750	.(3)14638	.(5)42709	0	0
3.4	.088017	.088804	.047263	.(2)55872	.(3)31068	.(4)10239	0	0
3.5	.10171	.10604	.060315	.(2)75665	.(3)44611	.(4)15585	.(6)36302	0
3.6	.11736	.12603	.076540	.010174	.(3)63500	.(4)23477	.(6)57866	0
3.8	.15579	.17580	.12141	.018041	.(2)12563	.(4)51788	.(5)14229	0
4.0	.20626	.24157	.18925	.031273	.(2)24164	.(3)11045	.(5)33640	0
4.2	.27281	.32765	.29065	.053162	.(2)45356	.(3)22876	.(5)76853	0
4.4	.36103	.43948	.44101	.088904	.(2)83376	.(3)46189	.(4)17039	.(6)45024
4.5	.41565	.50737	.54092	.11347	.011140	.(3)64580	.(4)24928	.(6)68908
4.6	.47862	.58374	.66253	.14661	.015053	.(3)91220	.(4)36798	.(5)10630
4.8	.63640	.76890	.98740	.23908	.026772	.(2)17679	.(4)77689	.(5)24442
5.0	.84962	1.0055	1.4626	.38613	.046994	.(2)33699	.(3)16076	.(5)54889

c	f_0	f_2	f_4	f_6	f_8	f_{10}	f_{12}	f_{14}
0	1.0000	0	0	0	0	0	0	0
0.1	1.0002	.(3)13340	.(8)90755	0	0	0	0	0
0.2	1.0008	.(3)53433	.(6)14546	0	0	0	0	0
0.4	1.0032	.(2)21494	.(5)23433	.(8)14578	0	0	0	0
0.5	1.0050	.(2)33726	.(5)57504	.(8)55926	0	0	0	0
0.6	1.0073	.(2)48818	.(4)12000	.(7)16815	0	0	0	0
0.8	1.0131	.(2)87945	.(4)38540	.(7)96259	0	0	0	0
1.0	1.0208	.013980	.(4)96073	.(6)37578	.(9)94089	0	0	0
1.2	1.0305	.020563	.(3)20440	.(5)11523	.(8)41607	0	0	0
1.4	1.0423	.028710	.(3)39046	.(5)30043	.(7)14789	0	0	0
1.5	1.0492	.033435	.(3)52349	.(5)46307	.(7)26193	0	0	0
1.6	1.0567	.038635	.(3)69034	.(5)69591	.(7)44830	0	0	0
1.8	1.0738	.050608	.(2)11521	.(4)14749	.(6)12050	.(9)68221	0	0
2.0	1.0941	.064975	.(2)18393	.(4)29180	.(6)29500	.(8)20651	0	0
2.2	1.1181	.082171	.(2)28367	.(4)54669	.(6)67038	.(8)56878	0	0
2.4	1.1463	.10275	.(2)42563	.(4)98033	.(5)14343	.(7)14507	0	0
2.5	1.1623	.11451	.(2)51693	.(3)12947	.(5)20259	.(7)22254	0	0
2.6	1.1797	.12739	.(2)62472	.(3)16961	.(5)29201	.(7)34724	0	0
2.8	1.2192	.15697	.(2)90075	.(3)28489	.(5)57036	.(7)78803	.(9)79682	0
3.0	1.2658	.19257	.012800	.(3)46681	.(4)10757	.(6)17092	.(8)19865	0
3.2	1.3212	.23554	.017972	.(3)74898	.(4)19689	.(6)35655	.(8)47207	0
3.4	1.3870	.28754	.024981	.(2)11801	.(4)35107	.(6)71887	.(7)10757	0
3.5	1.4244	.31754	.029353	.(2)14733	.(4)46466	.(5)10090	.(7)16009	0
3.6	1.4652	.35059	.034421	.(2)18299	.(4)61166	.(5)14062	.(7)23616	0
3.8	1.5583	.42713	.047061	.(2)27969	.(3)10436	.(5)26767	.(7)50131	.(9)7149
4.0	1.6691	.52004	.063878	.(2)42180	.(3)17467	.(5)49688	.(6)10318	.(8)1633
4.2	1.8006	.63269	.086102	.(2)62813	.(3)28709	.(5)90107	.(6)20640	.(8)3603
4.4	1.9564	.76902	.11526	.(2)92405	.(3)46384	.(4)15984	.(6)40193	.(8)7702
4.5	2.0447	.84745	.13301	.011158	.(3)58591	.(4)21120	.(6)55552	.(7)1113
4.6	2.1405	.93355	.15324	.013434	.(3)73713	.(4)27764	.(6)76308	.(7)1598
4.8	2.3572	1.1315	.20232	.019307	.(2)11531	.(4)47275	.(5)14144	.(7)3224
5.0	2.6115	1.3690	.26531	.027439	.(2)17766	.(4)78982	.(5)25637	.(7)6336

OBLATE SPHEROIDAL FUNCTIONS $m=1$, $\ell=0$

c	$-f_{-2}$	$+f_{-4}/\rho$	$+f_{-6}/\rho$	$+f_{-8}/\rho$	$+f_{-10}/\rho$	$+f_{-12}/\rho$	$+f_{-14}/\rho$	$+f_{-16}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(2)33373	.(5)27800	.(9)26480	0	0	0	0	0
0.2	.013134	.(4)43880	.(7)16727	.(11)33802	0	0	0	0
0.4	.050289	.(3)67670	.(5)10337	.(9)83628	0	0	0	0
0.5	.076146	.(2)16093	.(5)38466	.(8)48656	0	0	0	0
0.6	.10569	.(2)32371	.(4)11161	.(7)20346	.(10)22571	0	0	0
0.8	.17221	.(2)95316	.(4)58678	.(6)19055	.(9)37624	0	0	0
1.0	.24346	.021509	.(3)20805	.(5)10583	.(8)32700	.(11)67675	0	0
1.2	.31484	.041130	.(3)57680	.(5)42386	.(7)18892	.(10)56368	0	0
1.4	.38370	.070446	.(2)13556	.(4)13606	.(7)82718	.(9)33639	0	0
1.5	.41685	.089467	.(2)19849	.(4)22919	.(6)16041	.(9)74814	0	0
1.6	.44907	.11184	.(2)28364	.(4)37343	.(6)29723	.(8)15812	.(11)60148	0
1.8	.51092	.16831	.(2)54581	.(4)91369	.(6)[illegible]2286	.(8)62241	.(10)30001	0
2.0	.57001	.24392	.(2)98766	.(3)20515	.(5)25656	.(7)21402	.(9)12752	0
2.2	.62743	.34437	.017081	.(3)43164	.(5)65515	.(7)66259	.(9)47838	0
2.4	.68446	.47769	.028570	.(3)86414	.(4)15660	.(6)18888	.(8)16252	.(10)1048
2.5	.71336	.56043	.036917	.(2)12152	.(4)23935	.(6)31357	.(8)29299	.(10)2051
2.6	.74277	.65575	.046669	.(2)16665	.(4)35562	.(6)50446	.(8)51019	.(10)3865
2.8	.80399	.89573	.075003	.(2)31253	.(4)77610	.(5)12848	.(7)15093	.(9)13276
3.0	.86991	1.2232	.11931	.(2)57421	.(3)16424	.(5)30963	.(7)41817	.(9)42271
3.2	.94285	1.6768	.18883	.010402	.(3)33965	.(5)73454	.(6)113[illegible]3	.(8)13014
3.4	1.0252	2.3154	.29861	.018677	.(3)69060	.(4)16894	.(6)29389	.(8)38237
3.5	1.0708	2.7307	.37576	.024973	.(3)97995	.(4)25426	.(6)46900	.(8)66695
3.6	1.1198	3.2296	.47329	.033364	.(2)13870	.(4)38106	.(6)74408	.(7)10863
3.8	1.2301	4.5598	.75392	.059494	.(2)27626	.(4)84697	.(5)18446	.(7)30028
4.0	1.3600	6.5272	1.2090	.10613	.(2)54716	.(3)18610	.(5)44948	.(7)81125
4.2	1.5139	9.4792	1.9535	.18964	.010795	.(3)40516	.(4)10795	.(6)21490
4.4	1.6971	13.968	3.1815	.33965	.021239	.(3)87532	.(4)25604	.(6)55952
4.5	1.8297	19.936	4.9865	.58240	.038101	.(2)16426	.(4)50259	.(5)11488
4.6	1.9155	20.871	5.2204	.60972	.041684	.(2)18778	.(4)60035	.(5)14346
4.8	2.1759	31.592	8.6243	1.0966	.081599	.(2)40011	.(3)13924	.(5)36203
5.0	2.4862	48.379	14.330	1.9746	.15927	.(2)84673	.(3)31955	.(5)90110

c	f_1	f_3	f_5	f_7	f_9	f_{11}	f_{13}	f_{15}
0	1.0000	0	0	0	0	0	0	0
0.1	1.0003	.(3)12249	.(8)68739	0	0	0	0	0
0.2	1.0012	.(3)49047	.(6)11010	0	0	0	0	0
0.4	1.0049	.(2)19700	.(5)17693	.(9)92396	0	0	0	0
0.5	1.0077	.(2)30876	.(5)43335	.(8)35364	0	0	0	0
0.6	1.0111	.(2)44630	.(5)90218	.(7)10603	0	0	0	0
0.8	1.0199	.(2)80112	.(4)28803	.(7)60195	0	0	0	0
1.0	1.0314	.012674	.(4)71240	.(6)23271	.(9)49909	0	0	0
1.2	1.0457	.018531	.(3)15009	.(6)70628	.(8)21818	0	0	0
1.4	1.0630	.025682	.(3)28335	.(5)18156	.(8)76361	0	0	0
1.5	1.0728	.029778	.(3)37729	.(5)27759	.(7)13404	0	0	0
1.6	1.0835	.034253	.(3)49400	.(5)41363	.(7)22729	0	0	0
1.8	1.1075	.044396	.(3)81105	.(5)85989	.(7)59821	0	0	0
2.0	1.1353	.056295	.(2)12707	.(4)16641	.(6)14297	.(9)86952	0	0
2.2	1.1672	.070168	.(2)19181	.(4)30408	.(6)31619	.(8)23274	0	0
2.4	1.2037	.086276	.(2)28089	.(4)53016	.(6)65626	.(8)57498	0	0
2.5	1.2238	.095261	.(2)33663	.(4)68955	.(6)92628	.(8)88067	0	0
2.6	1.2453	.10493	.(2)40120	.(4)88905	.(5)12918	.(7)13285	0	0
2.8	1.2926	.12649	.(2)56122	.(3)14427	.(5)24317	.(7)29006	0	0
3.0	1.3462	.15139	.(2)77136	.(3)22766	.(5)44054	.(7)60329	.(9)61610	0
3.2	1.4069	.18014	.010444	.(3)35073	.(5)77219	.(6)12031	.(8)13979	0
3.4	1.4756	.21332	.013960	.(3)52916	.(4)13150	.(6)23128	.(8)30335	0
3.5	1.5133	.23179	.016071	.(3)64544	.(4)16996	.(6)31673	.(8)44019	0
3.6	1.5533	.25164	.018453	.(3)78391	.(4)21835	.(6)43045	.(8)63286	0
3.8	1.6413	.29591	.024157	.(2)11427	.(4)35450	.(6)77842	.(7)12749	0
4.0	1.7408	.34708	.031356	.(2)16421	.(4)56413	.(5)13720	.(7)24889	0
4.2	1.8535	.40627	.040395	.(2)23298	.(4)88172	.(5)23628	.(7)47236	0
4.4	1.9811	.47478	.051695	.(2)32675	.(3)13559	.(5)39848	.(7)87382	.(8)1485
4.5	2.0511	.51300	.058347	.(2)38543	.(3)16719	.(5)51374	.(6)11780	.(8)2093
4.6	2.1256	.55414	.065763	.(2)45352	.(3)20544	.(5)65932	.(6)15792	.(8)2931
4.8	2.2895	.64612	.083214	.(2)62353	.(3)30709	.(4)10721	.(6)27936	.(8)5643
5.0	2.4754	.75279	.10479	.(2)84987	.(3)45340	.(4)17153	.(6)48456	.(7)1061

c	$-f_{-1}$	$-f_{-3}/\rho$	$-f_{-5}/\rho$	$-f_{-7}/\rho$	$-f_{-9}/\rho$	$-f_{-11}/\rho$	$-f_{-13}/\rho$	$-f_{-15}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(3)66623	.(5)16665	.(9)37036	0	0	0	0	0
0.2	.(2)26597	.(4)26658	.(7)23702	.(11)75253	0	0	0	0
0.4	.010558	.(3)42622	.(5)15170	.(8)19271	0	0	0	0
0.5	.016405	.(2)10401	.(5)57875	.(7)11466	.(10)12031	0	0	0
0.6	.023464	.(2)21558	.(4)17286	.(7)49430	.(10)74698	0	0	0
0.8	.041026	.(2)68106	.(4)97247	.(6)49471	.(8)13295	0	0	0
1.0	.062818	.016640	.(3)37204	.(5)29596	.(7)12433	.(10)32530	0	0
1.2	.088385	.034598	.(2)11168	.(4)12805	.(7)77502	.(9)29208	0	0
1.4	.11730	.064460	.(2)28402	.(4)44374	.(6)36576	.(8)18769	0	0
1.5	.13288	.085304	.(2)43215	.(4)77551	.(6)73402	.(8)43247	0	0
1.6	.14918	.11103	.(2)64098	.(3)13095	.(5)14107	.(8)94582	0	0
1.8	.18378	.18045	.013230	.(3)34250	.(5)46725	.(7)3966	.(9)23040	0
2.0	.22091	.28076	.025501	.(3)81605	.(4)13753	.(6)14419	.(8)10343	0
2.2	.26053	.42253	.046602	.(2)18066	.(4)36864	.(6)46782	.(8)40614	0
2.4	.30275	.62004	.081666	.(2)37721	.(4)91649	.(5)13846	.(7)14309	.(9)10767
2.5	.32489	.74551	.10672	.(2)53514	.(3)14112	.(5)23137	.(7)25947	.(9)21187
2.6	.34779	.89276	.13845	.(2)75125	.(3)21432	.(5)38012	.(7)46110	.(9)40726
2.8	.396001	1.2676	.22864	.014401	.(3)47665	.(5)98064	.(6)13798	.(8)14135
3.0	.44784	1.7818	.36985	.026758	.(2)10169	.(4)24020	.(6)38801	.(8)45633
3.2	.50394	2.4883	.58873	.048474	.(2)20962	.(4)56335	.(5)10354	.(7)13854
3.4	.56506	3.4618	.92561	.086029	.(2)41991	.(3)12738	.(5)26426	.(7)39915
3.5	.59779	4.0806	1.1564	.11387	.(2)58889	.(3)18928	.(5)41607	.(7)66591
3.6	.63209	4.8093	1.4418	.15016	.(2)82136	.(3)27925	.(5)64934	.(6)10994
3.8	.70616	6.6861	2.2312	.25864	.015752	.(3)59641	.(4)15446	.(6)29131
4.0	.78847	9.3147	3.4371	.44077	.029714	.(2)12457	.(4)35731	.(6)74639
4.2	.88054	13.025	5.2807	.74495	.055292	.(2)25533	.(4)80689	.(5)18574
4.4	.98408	18.302	8.1037	1.2511	.10173	.(2)51498	.(3)17846	.(5)45057
4.5	1.0407	21.739	10.038	1.6182	.13749	.(2)72747	.(3)26356	.(5)69577
4.6	1.1010	25.860	12.435	2.0908	.18541	.010245	.(3)38761	.(4)10688
4.8	1.2338	36.775	19.103	3.4820	.33534	.020138	.(3)82865	.(4)24857
5.0	1.3851	52.656	29.398	5.7846	.60263	.039187	.(2)17472	.(4)56807

c	$-f_0$	f_2	f_4	f_6	f_8	f_{10}	f_{12}	f_{14}
0	0	1.00000	0	0	0	0	0	0
0.1	.(3)34288	0.99983	.(4)47662	0	0	0	0	0
0.2	.(2)13718	.99961	.(3)42313	.(7)8070	0	0	0	0
0.4	.(2)55004	.99845	.(2)16908	.(5)14937	0	0	0	0
0.5	.(2)86077	.99756	.(2)26397	.(5)31470	0	0	0	0
0.6	.012418	.99646	.(2)37976	.(5)65198	.(8)65766	0	0	0
0.8	.022180	.99360	.(2)67344	.(4)20558	.(7)37051	0	0	0
1.0	.034860	.98980	.010488	.(4)50034	.(6)14092	0	0	0
1.2	.050553	.98495	.015038	.(3)10335	.(6)41919	0	0	0
1.4	.069367	.97893	.020362	.(3)19055	.(5)10524	0	0	0
1.5	.079981	.97545	.023304	.(3)25042	.(5)15878	0	0	0
1.6	.091421	.97162	.026427	.(3)32318	.(5)23320	0	0	0
1.8	.11683	.96286	.033196	.(3)51415	.(5)46972	.(7)28636	0	0
2.0	.14574	.95250	.040619	.(3)77730	.(5)87701	.(7)66027	0	0
2.2	.17822	.94037	.048643	.(2)11275	.(4)15403	.(6)14037	0	0
2.4	.21436	.92634	.057205	.(2)15800	.(4)25707	.(6)27892	0	0
2.5	.23382	.91856	.061666	.(2)18494	.(4)32664	.(6)38467	0	0
2.6	.25419	.91027	.066236	.(2)21504	.(4)41099	.(6)52363	0	0
2.8	.29764	.89210	.075663	.(2)28545	.(4)63340	.(6)93661	0	0
3.0	.34458	.87180	.085420	.(2)37085	.(4)94589	.(5)16070	0	0
3.2	.39475	.84947	.095451	.(2)47291	.(3)13747	.(5)26601	.(7)36921	0
3.4	.44778	.82529	.10572	.(2)59349	.(3)19515	.(5)42684	.(7)66943	0
3.5	.47522	.81259	.11094	.(2)66139	.(3)23073	.(5)53519	.(7)88987	0
3.6	.50318	.79955	.11622	.(2)73478	.(3)27153	.(5)66687	.(6)11750	0
3.8	.56037	.77265	.12700	.(2)89948	.(3)37143	.(4)10183	.(6)19995	0
4.0	.61874	.74508	.13815	.010912	.(3)50106	.(4)15254	.(6)33240	0
4.2	.67772	.71735	.14982	.013146	.(3)66814	.(4)22483	.(6)54111	.(8)98029
4.4	.73686	.68993	.16220	.015759	.(3)88318	.(4)32714	.(6)86586	.(7)17240
4.5	.76639	.67647	.16875	.017232	.(2)10127	.(4)39301	.(5)10893	.(7)22703
4.6	.79588	.66322	.17556	.018831	.(2)11597	.(4)47105	.(5)13658	.(7)29772
4.8	.85475	.63752	.19020	.022467	.(2)15154	.(4)67277	.(5)21294	.(7)50632
5.0	.91365	.61295	.20645	.026797	.(2)19741	.(4)95485	.(5)32887	.(7)83790

c	f_{-2}	f_{-4}/ρ	f_{-6}/ρ	f_{-8}/ρ	f_{-10}/ρ	f_{-12}/ρ	f_{-14}/ρ	f_{-16}/ρ
0	0	0	0	0	0	0	0	0
0.1	.(6)19040	.(9)10579	0	0	0	0	0	0
0.2	.(5)30430	.(8)67641	0	0	0	0	0	0
0.4	.(4)48546	.(6)43194	.(8)14814	0	0	0	0	0
0.5	.(3)11824	.(5)16447	.(8)88147	0	0	0	0	0
0.6	.(3)24445	.(5)48992	.(7)37820	.(10)91712	0	0	0	0
0.8	.(3)76676	.(4)27358	.(6)37568	.(8)16200	0	0	0	0
1.0	.(2)18541	.(3)10352	.(5)22230	.(7)14983	.(10)53813	0	0	0
1.2	.(2)38001	.(3)30592	.(5)94720	.(7)91973	.(9)47578	0	0	0
1.4	.(2)69448	.(3)76157	.(4)32150	.(6)42516	.(8)29945	0	0	0
1.5	.(2)90830	.(2)11435	.(4)55478	.(6)84248	.(8)68131	0	0	0
1.6	.011664	.(2)16705	.(4)92324	.(5)15958	.(7)14687	.(10)85598	0	0
1.8	.018358	.(2)33226	.(3)23314	.(5)51050	.(7)59490	.(9)43896	0	0
2.0	.027440	.(2)61099	.(3)53149	.(4)14385	.(6)20708	.(8)18871	0	0
2.2	.039313	.010528	.(2)11142	.(4)36542	.(6)63698	.(8)70270	0	0
2.4	.054363	.017160	.(2)21768	.(4)85124	.(5)17675	.(7)23218	.(9)21189	0
2.5	.063188	.021506	.(2)29730	.(3)12629	.(5)28469	.(7)40592	.(9)40205	0
2.6	.072927	.026644	.(2)40035	.(3)18417	.(5)44932	.(7)69319	.(9)74279	0
2.8	.095272	.039599	.(2)69835	.(3)37367	.(4)10588	.(6)18961	.(8)23577	0
3.0	.12156	.056528	.011620	.(3)71640	.(4)23344	.(6)48039	.(8)68621	0
3.2	.15184	.077694	.018530	.(2)13057	.(4)48511	.(5)11373	.(7)18501	.(9)22184
3.4	.18600	.10298	.028420	.(2)22733	.(4)95607	.(5)25344	.(7)46590	.(9)63114
3.5	.20446	.11700	.034722	.(2)29528	.(3)13180	.(5)37056	.(7)72231	.(8)10374
3.6	.22376	.13177	.042060	.(2)37978	.(3)17964	.(5)53489	.(6)11038	.(8)16780
3.8	.26471	.16288	.060246	.(2)61115	.(3)32334	.(4)10752	.(6)24759	.(8)41981
4.0	.30832	.19461	.083768	.(2)95094	.(3)56004	.(4)20691	.(6)52889	.(8)99496
4.2	.35401	.22482	.11342	.014363	.(3)93763	.(4)38313	.(5)10820	.(7)22476
4.4	.40123	.25106	.15005	.021141	.(2)15242	.(4)68603	.(5)21315	.(7)48679
4.5	.42528	.26190	.17128	.025435	.(2)19246	.(4)90791	.(5)29546	.(7)70644
4.6	.44956	.27077	.19463	.030450	.(2)24164	.(3)11937	.(5)40649	.(6)10166
4.8	.49875	.28125	.24844	.043085	.(2)37527	.(3)20278	.(5)75420	.(6)20583
5.0	.54873	.27963	.31316	.060129	.(2)57330	.(3)33787	.(4)13681	.(6)40610

c	$-B_{2,0}$	f_0	f_2	f_4	f_6	f_8	f_{10}	f_{12}
0	6.———	1.———	0	0	0	0	0	0
0.1	5.99857	1.00010	.(4)40828	.(8)13748	0	0	0	0
0.2	5.99428	1.00041	.(3)16345	.(7)22022	0	0	0	0
0.4	5.97709	1.00164	.(3)65608	.(6)35393	.(9)13211	0	0	0
0.5	5.96416	1.00257	.(2)10278	.(6)86702	.(9)50589	0	0	0
0.6	5.94832	1.00370	.(2)14847	.(5)18053	.(8)15177	0	0	0
0.8	5.90777	1.00663	.(2)26611	.(5)57663	.(8)86299	0	0	0
1.0	5.85516	1.01044	.(2)42020	.(4)14271	.(7)33433	0	0	0
1.2	5.79015	1.01519	.(2)61300	.(4)30095	.(6)10175	.(9)24540	0	0
1.4	5.71225	1.02094	.(2)84741	.(4)56881	.(6)26243	.(9)86296	0	0
1.5	5.66831	1.02420	.(2)98132	.(4)75806	.(6)40207	.(8)15792	0	0
1.6	5.62092	1.02775	.011271	.(4)99327	.(6)60032	.(8)25834	0	0
1.8	5.51545	1.03570	.014564	.(3)16341	.(5)12542	.(8)68457	0	0
2.0	5.39501	1.04492	.018411	.(3)25670	.(5)24415	.(7)16492	0	0
2.2	5.25862	1.05552	.022877	.(3)38878	.(5)44927	.(7)36818	0	0
2.4	5.10509	1.06767	.028046	.(3)57173	.(5)78981	.(7)77251	0	0
2.5	5.02148	1.07437	.030924	.(3)68693	.(4)10321	.(6)10970	.(9)86374	
2.6	4.93304	1.08154	.034017	.(3)82089	.(4)13373	.(6)15399	.(8)13127	0
2.8	4.74081	1.09737	.040915	.(2)11557	.(4)21948	.(6)29406	.(8)29141	0
3.0	4.52646	1.11543	.048885	.(2)16008	.(4)35092	.(6)54163	.(8)61766	0
3.2	4.28770	1.13607	.058115	.(2)21880	.(4)54887	.(6)96741	.(7)12584	0
3.4	4.02185	1.15967	.068830	.(2)29578	.(4)84264	.(5)16831	.(7)24781	0
3.5	3.87780	1.17274	.074828	.(2)34268	.(3)10377	.(5)22008	.(7)34384	0
3.6	3.72575	1.18674	.081307	.(2)39623	.(3)12734	.(5)28627	.(7)47382	0
3.8	3.39575	1.21788	.095889	.(2)52684	.(3)18986	.(5)47749	.(7)88303	0
4.0	3.02762	1.25380	.11299	.(2)69631	.(3)27986	.(5)78303	.(6)16092	.(8)25294
4.2	2.61657	1.29538	.13315	.(2)91580	.(3)40846	.(4)12652	.(6)28742	.(8)49912
4.4	2.15721	1.34367	.15698	.011997	.(3)59108	.(4)20174	.(6)50441	.(8)96327
4.5	1.90760	1.37072	.17051	.013714	.(3)70900	.(4)25362	.(6)66416	.(7)13280
4.6	1.64367	1.39992	.18526	.015665	.(3)84892	.(4)31794	.(6)87116	.(7)18219
4.8	1.06971	1.46559	.21892	.020397	.(2)12110	.(4)49568	.(5)14827	.(7)33826
5.0	0.42896	1.54238	.25907	.026490	.(2)17164	.(4)76505	.(5)24890	.(7)61723

c	f_{-2}	$-f_{-4}$	$-f_{-6}/\rho$	$-f_{-8}/\rho$	$-f_{-10}/\rho$	$-f_{-12}/\rho$	$-f_{-14}/\rho$	$-f_{-16}/\rho$
0	0	0	0	0	0	0	0	0
0.1	-.(2)13271	.(5)66545	0	0	0	0	0	0
0.2	-.(2)52344	.(3)10590	.(7)47126	0	0	0	0	0
0.4	-.019774	.(2)16577	.(5)29620	0	0	0	0	0
0.5	-.029568	.(2)39804	.(4)11145	0	0	0	0	0
0.6	-.040300	.(2)80863	.(4)32719	.(7)46948	0	0	0	0
0.8	-.061878	.024238	.(3)17593	.(6)45032	0	0	0	0
1.0	-.078817	.055210	.(3)63353	.(5)25450	0	0	0	0
1.2	-.085744	.10508	.(2)17616	.(4)10245	.(7)41654	0	0	0
1.4	-.078619	.17588	.(2)40836	.(4)32535	.(6)18065	0	0	0
1.5	-.069058	.21911	.(2)58974	.(4)54132	.(6)34567	0	0	0
1.6	-.055376	.26724	.(2)82710	.(4)86710	.(6)63125	0	0	0
1.8	-.016010	.37664	.015104	.(3)20210	.(5)18703	0	0	0
2.0	+.037829	.50022	.025440	.(3)42424	.(5)48705	.(7)38809	0	0
2.2	.10353	.63375	.040205	.(3)81979	.(4)11449	.(6)11075	0	0
2.4	.17824	.77342	.060427	.(2)14832	.(4)24797	.(6)28647	0	0
2.5	.21815	.84460	.072949	.(2)19549	.(4)35572	.(6)44674	0	0
2.6	.25937	.91630	.087304	.(2)25467	.(4)50282	.(6)68435	0	0
2.8	.34491	1.0606	.12232	.(2)41944	.(4)96697	.(5)15325	0	0
3.0	.43357	1.2056	.16740	.(2)66855	.(3)17820	.(5)32562	0	0
3.2	.52474	1.3514	.22510	.010388	.(3)31743	.(5)66293	.(7)99833	0
3.4	.61850	1.4992	.29893	.015829	.(3)55041	.(4)13038	.(6)22236	0
3.5	.66651	1.5744	.34332	.019430	.(3)71885	.(4)18089	.(6)32742	0
3.6	.71546	1.6507	.39375	.023780	.(3)93468	.(4)24944	.(6)47846	0
3.8	.81679	1.8084	.51639	.035376	.(2)15624	.(4)46690	.(5)10012	0
4.0	.92419	1.9753	.67656	.052316	.(2)25824	.(4)85942	.(5)20488	0
4.2	1.0399	2.1551	.88826	.077174	.(2)42367	.(3)15624	.(5)41202	.(7)81471
4.4	1.1667	2.3526	1.1718	.11390	.(2)69226	.(3)28159	.(5)81768	.(6)17787
4.5	1.2354	2.4596	1.3495	.13852	.(2)88437	.(3)37722	.(4)11476	.(6)26141
4.6	1.3082	2.5731	1.5573	.16864	.011298	.(3)50479	.(4)16073	.(6)38300
4.8	1.4687	2.8233	2.0891	.25103	.018465	.(3)90255	.(4)31387	.(6)81615
5.0	1.6533	3.1111	2.8343	.37635	.030275	.(2)16129	.(4)61037	.(5)17256

c	f_1	f_3	f_5	f_7	f_9	f_{11}	f_{13}	f_{15}
0	1.00000	0	0	0	0	0	0	0
0.1	1.00019	.(4)52923	.(8)16823	.(13)35524	0	0	0	0
0.2	1.00074	.(3)21185	.(7)26938	.(11)22755	0	0	0	0
0.4	1.00287	.(3)84985	.(6)43243	.(9)14614	0	0	0	0
0.5	1.00465	.(2)13308	.(5)10583	.(9)55895	0	0	0	0
0.6	1.00671	.(2)19251	.(5)22011	.(8)16744	.(12)90684	0	0	0
0.8	1.01198	.(2)34393	.(5)70101	.(8)94849	.(11)91357	0	0	0
1.0	1.01884	.(2)54212	.(4)17284	.(7)36564	.(10)55053	0	0	0
1.2	1.02733	.(2)78911	.(4)36274	.(6)11059	.(9)23991	0	0	0
1.4	1.03754	.010879	.(4)68169	.(6)28315	.(9)83657	0	0	0
1.5	1.04332	.012578	.(4)90554	.(6)43200	.(8)14657	.(11)37186	0	0
1.6	1.04955	.014421	.(3)11823	.(6)64210	.(8)24797	.(11)71597	0	0
1.8	1.06347	.018562	.(3)19297	.(5)13279	.(8)64956	.(10)23751	0	0
2.0	1.07941	.023354	.(3)30039	.(5)25554	.(7)15445	.(10)69766	0	0
2.2	1.09753	.028860	.(3)45020	.(5)46405	.(7)33970	.(9)18580	0	0
2.4	1.11800	.035153	.(3)65420	.(5)80372	.(7)70090	.(9)45655	0	0
2.5	1.12917	.038621	.(3)78089	.(4)10418	.(7)98632	.(9)69738	0	0
2.6	1.14101	.042319	.(3)92670	.(4)13383	.(6)13711	.(8)10490	0	0
2.8	1.16679	.050458	.(2)12849	.(4)21556	.(6)25643	.(8)22770	.(10)15676	0
3.0	1.19559	.059685	.(2)17497	.(4)33755	.(6)46148	.(8)47081	.(10)37233	0
3.2	1.22773	.070136	.(2)23462	.(4)51588	.(6)80342	.(8)93338	.(10)84035	0
3.4	1.26353	.081969	.(2)31047	.(4)77203	.(5)13589	.(7)17838	.(9)18142	0
3.5	1.28292	.088459	.(2)35558	.(4)93782	.(5)17503	.(7)24357	.(9)26260	0
3.6	1.30338	.095365	.(2)40616	.(3)11343	.(5)22411	.(7)33009	.(9)37662	0
3.8	1.34774	.11054	.(2)52611	.(3)16400	.(5)36145	.(7)59367	.(9)75518	0
4.0	1.39710	.12773	.(2)67560	.(3)23376	.(5)57151	.(6)10410	.(8)14681	.(10)16510
4.2	1.45206	.14723	.(2)86101	.(3)32900	.(5)88781	.(6)17842	.(8)27759	.(10)34433
4.4	1.51327	.16937	.010900	.(3)45786	.(4)13574	.(6)29963	.(8)51190	.(10)69716
4.5	1.54644	.18155	.012237	.(3)53806	.(4)16694	.(6)38555	.(8)68916	.(10)98190
4.6	1.58148	.19453	.013719	.(3)63078	.(4)20460	.(6)49393	.(8)92281	.(9)13741
4.8	1.65755	.22316	.017177	.(3)86112	.(4)30440	.(6)80065	.(7)16295	.(9)26428
5.0	1.74244	.25577	.021408	.(2)11659	.(4)44755	.(5)12780	.(7)28234	.(9)48700

OBLATE SPHEROIDAL FUNCTIONS m=2, ℓ=1

c	$-f_{-1}$	$+f_{-3}$	$-f_{-5}/\rho$	$-f_{-7}/\rho$	$-f_{-9}/\rho$	$-f_{-11}/\rho$	$-f_{-13}/\rho$	$-f_{-15}/\rho$
0	0	0	0	0	0	0	0	0
0.1	.(3)57230	.(6)95172	.(9)52890	0	0	0	0	0
0.2	.(2)22996	.(4)15196	.(7)33812	0	0	0	0	0
0.4	.(2)93621	.(3)24109	.(5)21540	.(8)24651	0	0	0	0
0.5	.014817	.(3)58489	.(5)81884	.(7)14654	0	0	0	0
0.6	.021663	.(2)12035	.(4)24348	.(7)62804	0	0	0	0
0.8	.039944	.(2)37304	.(3)13538	.(6)62231	0	0	0	0
1.0	.065143	.(2)88845	.(3)50966	.(5)36713	0	0	0	0
1.2	.098306	.017879	.(2)14982	.(4)15602	.(7)91298	0	0	0
1.4	.14051	.031990	.(2)37115	.(4)52841	.(6)42172	0	0	0
1.5	.16532	.041356	.(2)55611	.(4)91111	.(6)83569	0	0	0
1.6	.19276	.052464	.(2)81100	.(3)15157	.(5)15837	0	0	0
1.8	.25595	.080449	.016103	.(3)38307	.(5)50788	0	0	0
2.0	.33081	.11695	.029658	.(3)87652	.(4)14388	.(6)14889	0	0
2.2	.41793	.16278	.051422	.(2)18516	.(4)36898	.(6)46284	0	0
2.4	.51775	.21861	.084882	.(2)36644	.(4)87154	.(5)13035	0	0
2.5	.57252	.25044	.10738	.(2)50497	.(3)13058	.(5)21213	0	0
2.6	.63059	.28494	.13459	.(2)68734	.(3)19258	.(5)33872	0	0
2.8	.75678	.36219	.20640	.012328	.(3)40208	.(5)82188	0	0
3.0	.89667	.45072	.30790	.021298	.(3)80049	.(4)18825	.(6)30341	0
3.2	1.0507	.55095	.44897	.035662	.(2)15311	.(4)41057	.(6)75399	0
3.4	1.2197	.66339	.64256	.058168	.(2)28307	.(4)85885	.(5)17831	0
3.5	1.3100	.72434	.76415	.073659	.(2)38063	.(3)12252	.(5)26975	0
3.6	1.4044	.78864	.90572	.092817	.(2)50847	.(3)17335	.(5)40407	0
3.8	1.6061	.92761	1.2613	.14544	.(2)89139	.(3)33937	.(5)88266	0
4.0	1.8266	1.0815	1.7400	.22453	.015310	.(3)64728	.(4)18680	.(6)3918
4.2	2.0680	1.2518	2.3836	.34247	.025848	.(2)12074	.(4)38467	.(6)8902
4.4	2.3330	1.4405	3.2493	.51737	.043020	.(2)22100	.(4)77376	.(5)1967
4.5	2.4752	1.5424	3.7887	.63400	.055243	.(2)29713	.(3)10888	.(5)2896
4.6	2.6247	1.6499	4.4157	.77573	.070757	.(2)39806	.(3)15251	.(5)4240
4.8	2.9472	1.8830	5.9929	1.1567	.11527	.(2)70734	.(3)29540	.(5)8994
5.0	3.3052	2.1432	8.1338	1.7180	.18634	.012427	.(3)56366	.(4)1853

c	f_0	f_2	f_4	f_6	f_8	f_{10}	f_{12}
0	1.0000	0	0	0	0	0	0
0.1	1.0001	.(4)17640	.(9)33645	0	0	0	0
0.2	1.0002	.(4)70602	.(8)53875	0	0	0	0
0.4	1.0010	.(3)28307	.(7)86476	.(10)20883	0	0	0
0.5	1.0015	.(3)44307	.(6)21163	.(10)79886	0	0	0
0.6	1.0022	.(3)63940	.(6)44012	.(9)23935	0	0	0
0.8	1.0040	.(2)11430	.(5)14014	.(8)13566	0	0	0
1.0	1.0063	.(2)17986	.(5)34546	.(8)52333	0	0	0
1.2	1.0091	.(2)26127	.(5)72488	.(7)15843	.(10)26797	0	0
1.4	1.0125	.(2)35934	.(4)13620	.(7)40610	.(10)93636	0	0
1.5	1.0144	.(2)41493	.(4)18091	.(7)62000	.(9)16425	0	0
1.6	1.0164	.(2)47507	.(4)23619	.(7)92222	.(9)27823	0	0
1.8	1.0210	.(2)60966	.(4)38548	.(6)19106	.(9)73099	0	0
2.0	1.0263	.(2)76458	.(4)60008	.(6)36841	.(8)17441	0	0
2.2	1.0323	.(2)94155	.(4)89956	.(6)67071	.(8)38514	0	0
2.4	1.0390	.011426	.(3)13078	.(5)11651	.(8)79838	0	0
2.5	1.0429	.012532	.(3)15619	.(5)15131	.(7)11267	.(10)65701	0
2.6	1.0465	.013703	.(3)18540	.(5)19469	.(7)15703	.(10)99153	0
2.8	1.0550	.016273	.(3)25736	.(5)31492	.(7)29551	.(9)21690	0
3.0	1.0644	.019173	.(3)35101	.(5)49558	.(7)53566	.(9)45241	0
3.2	1.0749	.022442	.(3)47170	.(5)76182	.(7)94026	.(9)90589	0
3.4	1.0867	.026128	.(3)62598	.(4)11479	.(6)16054	.(8)17509	0
3.5	1.0931	.028146	.(3)71818	.(4)13997	.(6)20786	.(8)24057	0
3.6	1.0998	.030291	.(3)82195	.(4)17000	.(6)26763	.(8)32817	0
3.8	1.1144	.034998	.(2)10696	.(4)24806	.(6)43696	.(8)59879	0
4.0	1.0308	.040332	.(2)13815	.(4)35737	.(6)70060	.(7)10671	0
4.2	1.1491	.046395	.(2)17731	.(4)50922	.(5)11056	.(7)18627	.(9)24974
4.4	1.1697	.053308	.(2)22640	.(4)71876	.(5)17209	.(7)31926	.(9)25422
4.5	1.1810	.057127	.(2)25542	.(4)85128	.(5)21370	.(7)41539	.(9)64176
4.6	1.1930	.061216	.(2)28788	.(3)10063	.(5)26463	.(7)53843	.(9)87035
4.8	1.2192	.070299	.(2)36483	.(3)13993	.(5)40266	.(7)89519	.(8)15797
5.0	1.2490	.080772	.(2)46118	.(3)19343	.(5)60703	.(6)14696	.(8)28214

c	f_{-2}	f_{-4}	f_{-6}	f_{-8}/ρ	f_{-10}/ρ	f_{-12}/ρ	f_{-14}/ρ
0	0	0	0	0	0	0	0
0.1	-.(3)85985	.(5)28514	-.(8)31706	.(12)11325	0	0	0
0.2	-.(2)34717	.(4)38233	-.(6)17043	.(10)24367	0	0	0
0.4	-.014375	.(3)70814	-.(4)12740	.(8)73038	0	0	0
0.5	-.022996	.(2)16969	-.(4)48026	.(7)43098	0	0	0
0.6	-.033990	.(2)34420	-.(3)14145	.(6)18320	.(9)13368	0	0
0.8	-.063849	.010268	-.(3)76659	.(5)17752	.(8)23091	0	0
1.0	-.10497	.023248	-.(2)27902	.(4)10171	.(7)20742	0	0
1.2	-.15702	.043875	-.(2)78613	.(4)41641	.(6)12281	.(9)30468	0
1.4	-.21759	.072512	-.018488	.(3)13475	.(6)54358	.(8)18408	0
1.5	-.24964	.089533	-.026888	.(3)22636	.(5)10511	.(8)40929	0
1.6	-.28180	.10799	-.037956	.(3)36595	.(5)19393	.(8)86063	0
1.8	-.34247	.14746	-.070019	.(3)86698	.(5)58530	.(7)32997	0
2.0	-.39076	.18669	-.11838	.(2)18398	.(4)15448	.(6)10797	0
2.2	-.41741	.22052	-.18608	.(2)35651	.(4)36516	.(6)31025	.(8)20268
2.4	-.41409	.24375	-.27496	.(2)64001	.(4)78732	.(6)80011	.(8)62406
2.5	-.39923	.24998	-.32757	.(2)83679	.(3)11222	.(5)12408	.(7)10519
2.6	-.37463	.25186	-.38535	.010774	.(3)15707	.(5)18837	.(7)17303
2.8	-.29577	.24163	-.51607	.017167	.(3)29338	.(5)41049	.(7)43896
3.0	-.17725	.21144	-.66480	.026110	.(3)51819	.(5)83769	.(6)10324
3.2	-.021274	.16117	-.82854	.0 8186	.(3)87309	.(4)16169	.(6)22770
3.4	.16832	.091849	-1.0041	.054040	.(2)14136	.(4)29771	.(6)47544
3.5	.27427	.050588	-1.0955	.063623	.(2)17761	.(4)39787	.(6)67489
3.6	.38692	.(2)52408	-1.1888	.074444	.(2)22146	.(4)52692	.(6)94786
3.8	.63018	-.096686	-1.3804	.10030	.(2)33752	.(4)90206	.(5)18170
4.0	.89440	-.21206	-1.5775	.13277	.(2)50307	.(3)15026	.(5)33708
4.2	1.1771	-.33944	-1.7798	.17335	.(2)73657	.(3)24472	.(5)60849
4.4	1.4771	-.47796	-1.9877	.22405	.010636	.(3)39145	.(4)10741
4.5	1.6336	-.55132	-2.0942	.25399	.012729	.(3)49235	.(4)14170
4.6	1.7947	-.62746	-2.2026	.28753	.015200	.(3)61736	.(4)18619
4.8	2.1316	-.78849	-2.4264	.36745	.021567	.(3)96326	.(4)31813
5.0	2.4918	-.96266	-2.6627	.46890	.030471	.(2)14918	.(4)53766